SHADES AND SHADOWS

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SHADES AND SHADOWS

AN EXPOSITION AND DEMONSTRATION OF SHORT AND CONVENIENT METHODS FOR DETERMINING THE SHADES AND SHADOWS OF OBJECTS ILLUMINATED BY THE CONVENTIONAL PARALLEL RAYS; THE METHODS IN USE AT THE ECOLE DES BEAUX ARTS AT PARIS

FROM THE FRENCH OF

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PREFACE.

THIS translation has been undertaken for the convenience of the students and for the advancement of the courses of the Department of Architecture of the University of Pennsylvania.

The ordinary methods of Descriptive Geometry as commonly applied in Casting Shadows are not well adapted to the hurried conditions of Architectural practice. The result is that shadows are ordinarily guessed at merely because of insufficient time to work them out by a laborious method. Under such circumstances only those who have studied the subject with great care and much practice can avoid gross and misleading errors.

The value of Shadows on Elevations is only beginning to be understood among American draftsmen. Shadows show the projection of the different parts, profiles of moldings, pitch of roofs, reveal of windows and doors, and many other details which could not otherwise be understood without a perspective drawing.

M. Pillet has succeeded admirably in simplifying the practice of casting shadows:

1st.—By avoiding the use of more than the one plane of projection in which the shadows are to appear.

2d.—By formulating simple and easily remembered rules for the Shades and Shadows of the more common forms.

It has been the aim of the Translator to keep closely to the original; the only deviation from the aim has been for the purpose of further elucidating points which might give trouble to readers not familiar with Descriptive Geometry. In many places explanations have been added in foot-notes, and every effort has been made to place the subject within the grasp of any intelligent Draftsman.

As a few Trigonometrical formulæ have been found necessary a short explanation of the principles of Trigonometry has been inserted in a foot-note under § 4.

The original work was intended for the use of Mechanical and Civil Engineers, as well as Architects, and included several problems which I have considered unnecessary, since shadows are seldom employed by Engineers in this country.

M. Pillet's method for pediments has been replaced by another which appears to me to be somewhat simpler.

The drawings I have carefully redrawn at large scale, and have added several.

The kind assistance of Dr. I. J. Schwatt, of the Department of Mathematics, in reading the manuscript, is gratefully acknowledged.

JULIAN MILLARD.

PHILADELPHIA, October, 1896.

CONVENTIONAL SHADES AND SHADOWS.

CHAPTER I.

GENERAL PRINCIPLES.

§ 1. Definitions.—The theory of Shades and Shadows has for its object, knowing the positions of one or more sources of light, to determine on the surfaces of bodies the portions of those surfaces which are in light and the portions which are not in light. The latter are said to be in shade or in shadow.

A ray of light is propagated in a straight line.

When a body is subjected to rays which emanate from a given source of light the rays of light are classed, with reference to the body, in three groups.

First. Rays *incident*, which are arrested by the surface of the object and illuminate it more or less according to their angle of incidence with the surface.

Second. Rays *lateral*, which pass beside the object and go on to illuminate others.

Third. Rays *tangential*, which fill the transition between the two preceding.

These last are the most important for our consideration.

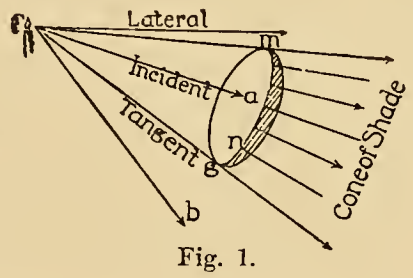

Fig. 1.

If the surface is rounded these rays are tangent to it and form a cone of shade in the case of a light at a finite distance, and a cylinder of shade in the case of a light at an infinite distance.

The line of contact of the cone with the object, mng (Fig. 1), is called the separatrix of shade and light. The part unlighted is called the shade of the surface.

If the surface is limited by a linear contour, as a disc or a polyhedron, the rays are no longer tangents according to the rigorous acceptance of the word. They pass through the limiting line of the contour or through certain edges of the polyhedron, to be determined.

The problems of Shades and Shadows, of Perspective and of Projections are almost identical. If we imagine at the source of light a "Point of Sight," the rays of light become visual rays.

The tangent rays which determine the separatrix become the visual rays of the contour; prolong these to a surface (plane or otherwise), and stop them there. In the first case they give the shadow of the first surface on the second taken as a screen; in the second case they give the image of the first on the second taken as a picture plane; the third case is like the others excepting that the source of light is taken at an infinite distance. (See Fig. 2.)

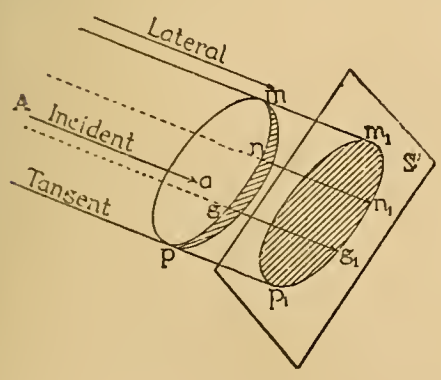

Fig. 2.

The problem of shadows in the case of sunlight is identical with that of oblique projections, since the source of light is at an infinite distance and the plane receiving the light is not at right angles to the rays of light.

The shadow of an object on a plane is nothing more than the oblique projection of the object on that plane.

§ 2. The Three General Methods of Finding Shadows.

First. *Slicing method.* (Fig. 3.)—Assume two surfaces S and S′ in light. Cut both surfaces by planes which are parallel to rays of light, the sun being the source of light.

We thus obtain two auxiliary curves; one in each surface.

Draw rays tangent to each of these curves at a and b on the first curve, and at m and n on the second; these are points of the separatrix.

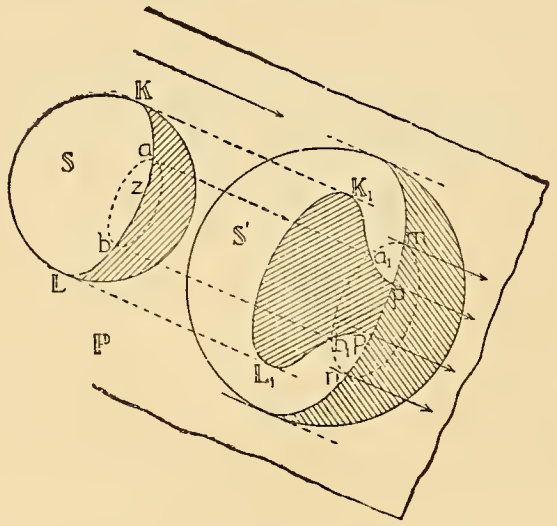

Fig. 3.

The first tangents, through a and b, if prolonged, intersect the second curve in a_1 and b_1 and are points of the shadow of the first surface on the second.

Repeat the operation by making as many slices as are necessary and join points of the same kind by continuous curves; we then obtain in KabL and in mppn the two separatrices and in $K_1a_1ppb_1L_1$ the shadow cast by S on S′.

The points p and p, where the shadow is lost in the shade, are called "points of loss." They do not always exist. (See Fig. 4.) In the case of Figure 4 the shadow is called "*closed*," and in Figure 3 it is called "*open*."

This method is in general one of difficult application. It is very convenient, however, in finding particular points of the line of shadow.

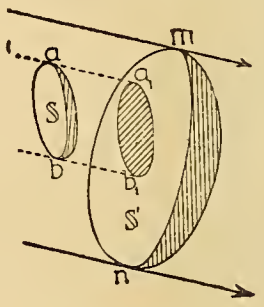

Fig. 4.

SECOND. *Method of circumscribing surfaces.*—Assume a surface (a sphere for example) of which the separatrix is to be determined. (Fig. 5.) Circumscribe about it a more simple surface, a cone for example, the separatrix of which is supposed to be known (a straight line in the case of a cone). The point m, where this straight line intersects the curve of contact ab of the cone and the surface, is a point of the separatrix of the latter (evident).

We may take a cylinder tangent to the sphere along the curve cd, which gives in n another point of the separatrix. Joining hmn and g by a continuous curve we have the required separatrix.

NOTE.—Except for certain particular cases, the separatrix Sm of the auxiliary circumscribing surface is not tangent to the line hmng of the given surface. In other words, when two surfaces are tangent their lines of shade are not always tangent. The lines of shade are tangent only when the contact of the two surfaces is what is called osculatory. (The contact of two lines is osculatory when, at the point of contact, the radius of curvature of the two lines is the same.)

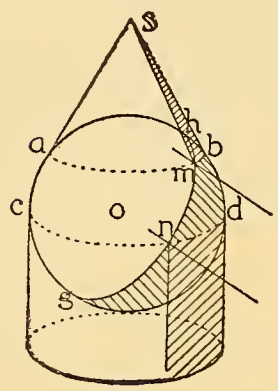

Fig. 5.

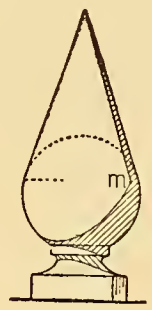

Fig. 6.

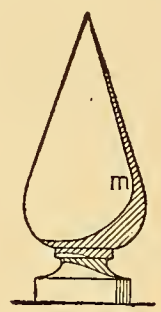

Fig. 7.

The contact will be osculatory if the profile is drawn free-hand (Fig. 7), as is generally done in architecture.

THIRD. *Method of auxiliary shadows or method of oblique projections.*—Assume two surfaces, S and S′ (Fig. 8), upon which it is desired to perform all the operations in shades and shadows indicated below.

(a) *Shade*[1] *on the surface S.*

1st. On the surface S draw a series of curves, No. 1, No. 2, No. 3, conveniently chosen.

In choosing these sections two things are essential:

(a) The lines of section should be of simple form, so as to be easily cast on the Auxiliary Plane; (b) they should all be of the same form.

2d. Take arbitrarily an auxiliary surface Z, ordinarily a plane, upon which are found in No. 1′, No. 2′, No. 3′, the shadows of the above curves.

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[1] Throughout the course on Shades and Shadows a distinction in meaning will be observed between the word *shade* and the word *shadow*.

That portion of a body which is darkened by being turned away from the source of light, and which is bounded by the separatrix, will be referred to as the *shade* of that body; while that portion of a body which is darkened by the interposition of another body, or another part of the same body, between it and the source of light will be referred to as the *shadow* of the second body on the first.

It is easily demonstrated, as it may also be done for the contour and the chosen curves, that the total shadow of S is the envelope[1] of the shadows Nos. 1′, 2′, 3′ cast by the chosen curves.

3d. Then draw this envelope tangent to the auxiliary shadows Nos. 1′, 2′, and 3′; this will be the total shadow cast by the surface S on the plane Z.

4th. Determine the points of contact a_1b_1–c_1d_1, etc., of the total shadow with the curves Nos. 1′, 2′, and 3′, and passing back along the rays from these points to the original curves, we obtain in abcd, etc., points of the separatrix of the surface S.

(b) *Shadow cast by the surface S′ on the surface S.*

FIRST. Find, as above, the total shadow of S′ on the auxiliary plane. (The figure does not indicate the construction by which this is effected.)

SECOND. Take the points of intersection m′, n′, etc., of this second shadow with the partial shadows Nos. 1′, 2′, 3′, and, passing back along rays from these points to the original curves, we find m, n, etc., points of the required shadow.

(c) *Points of loss.* The points p′ and q′, where the total shadow of S on the plane Z is cut by the shadow of S′, enables us, as shown by the figure, to find the points of loss p and q on the surface S by passing back along the rays.

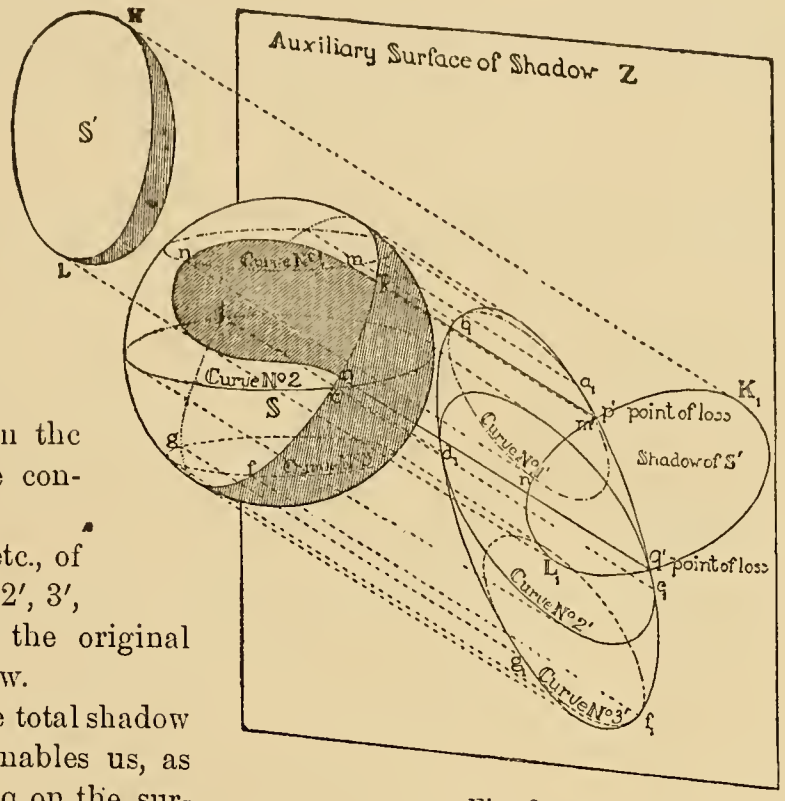

Fig. 8.

We shall demonstrate in § 3, Theorem 2, that in these points of loss, p and q, the curve of shadow, pmnq, must be tangent at p and q to rays of light.

REMARKS.—1st. This method is general; it is the one whose application is ordinarily least difficult. The auxiliary plane, instead of being taken arbitrarily, may sometimes be found in the same drawing.

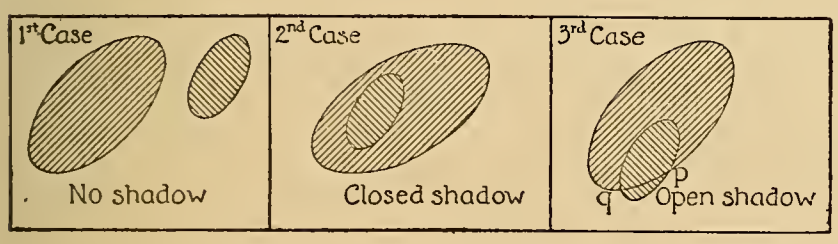

Fig. 9.

In place of a plane we may take any other auxiliary—as a cone, a cylinder, etc. In practice there are generally surfaces represented in the drawing which may serve as auxiliary planes for the application of the method of Oblique Projections.

2d. From the appearance of auxiliary shadows (Fig. 9) we observe three different cases of shadows. In the first case the surface S′ casts no shadow on the surface S; in the second the shadow is closed, without points of loss; in the third the shadow is open, and there are two points of loss corresponding to p′ and q′.

§ 3. THE THREE GENERAL THEOREMS.

Theorem 1. (Theorem of contours.)

THE CONTOUR of a surface, its separatrix and its curve of shadow on any other surface, have, in projection or in perspective, rays of light which are tangent common to all three.

This theorem may be applied in Figs. 3, 4, and 8, and also in Fig. 10 herewith.

a a_1, drawn tangent to the contour of S, is also the line of contour of the cone of shade, and it will also appear tangent to any section of the cone; and as the shadow on S′ is a section of the cone the line will also appear tangent to it.

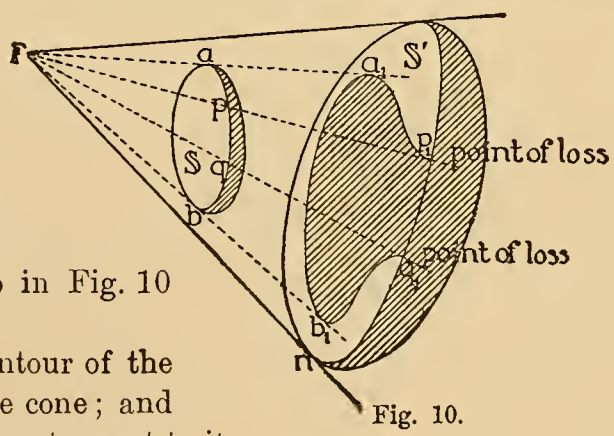

Fig. 10.

[1] The envelope of a series of curves is another curve tangent to each of the series.

Notice that at a_1 and b_1 the line of shadow and the ray of light are not in reality tangent; the tangency is only produced in the projection or in the perspective. The same is true at a and b, the ray of light appearing tangent to the separatrix and at the same time to the contour of the surface S.

Theorem 2. (Theorem of points of loss, Fig. 10.)

At any point of loss, the ray of light is tangent to the line of shadow, not only in projection, but also in space.

The line p p_1 is a generatrix of the cone of shade and is tangent to the surface of S′ at p_1. The line of shadow is a partial section of the cone, the portion of the cone between p_1 and q_1 being uncut.

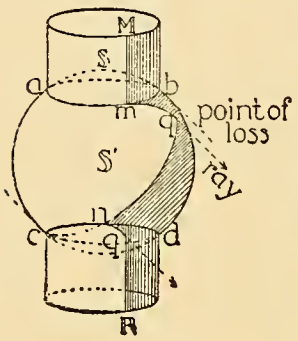

Fig. 11.

It is evident that the curve a_1p_1, as it approaches p_1, more and more nearly approaches to the direction of a generatrix of the cone, and at p_1 becomes tangent to the generatrix p p_1.

This theorem is subject to exception only in the case when two points of loss approach one another so closely as to coincide. In this case a line passing through the point of loss at right angles to the ray of light and tangent to the surface S′, will be tangent to the line of shadow.[1]

Theorem 3. (Theorem of intersecting surfaces, Fig. 11.)

When two surfaces intersect in a curve the shadow from the first on the second begins from the point where the separatrix of the first cuts the intersection, and at that point it is tangent to the line of intersection.

At m the tangent to the curve of intersection ab is the intersection of planes tangent to the two surfaces since it must lie in both of those planes.

The tangent to the line of shadow at this same point m is the intersection of the plane of shade from the first surface (also tangent to that surface) with a plane through m tangent to the second. Now these two planes, identical with the planes mentioned above, give in their intersection the tangent to the curve ab and at the same time that to the curve mq. These two tangents are one and the same line.

These three theorems enable us to draw the lines of shade and shadow with great accuracy and in their true direction by the use of but few points.

§ 4. THE 45° RAY OF LIGHT AND ITS REDUCTION TO THE ANGLE Φ.

In geometric drawing the luminous rays are considered as emanating from the sun. They are, then, parallel; and, furthermore, are chosen parallel to the diagonals of a cube (Fig. 12), two of whose faces are parallel with the planes of projection and whose other faces are in profile. This diagonal is called the Empirical Ray.

This ray has for its principal advantage the fact that it makes equal angles with the three planes of projection upon which are drawn the plans, elevations, and sections of objects. We designate this angle by the Greek letter Phi (Φ).

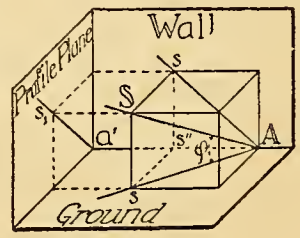

Fig. 12.

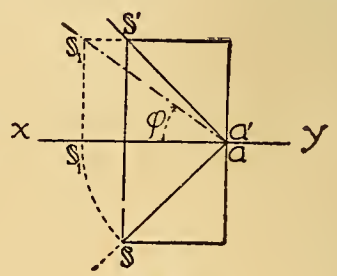

Fig. 13.

The projections of the Empirical Ray (Fig. 13) make 45° angles with the ground line; but it is important to note that the ray does not make angles of 45° with the planes of projection. In order to find these angles in their true magnitude it is necessary to make the following reduction:

If we reduce the luminous ray at a′ upon the vertical plane, by revolving S into the vertical plane on a vertical axis through a (Fig. 13), it then makes with the ground line the angle Φ.

Now Tan. Φ = Sin. 45°, as may be seen from Fig. 13a[2], hence Tan. Φ = .7071; and the angle Φ may

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[1] This will appear evident when we conceive of the shadow as the intersection of the cone with the surface S′.

[2] Those having no knowledge of Trigonometry will find in the following remarks all the information on that subject necessary for an intelligent understanding of the problems taken up in this work. Trigonometry

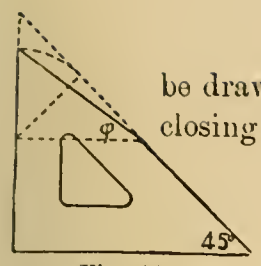

Fig. 14.

be drawn by laying off a line 1 in length and another at right angles about .707 in length; the closing side forms the angle Φ with the first line.

Triangle of angle Φ. Fig. 14 represents a triangle with angle Φ, which is very useful in finding conventional Shades and Shadows *and will be often used in the work to follow.*

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treats of the measurement of triangles, and consequently must deal with angles and lines and their relations to one another. By Geometry the sum of the angles of any triangle is 180°. Then in a right angled triangle the sum of the two acute angles is 90°, and either one is the "complement" of the other.

In Fig. 13a let Oba be any right triangle, with any angle $\propto$ at O, whose hypothenuse Oa is 1 in length. Now evidently the proportion of ab to Oa varies with the angle $\propto$ and Ob varies with the angle at a. But the angle at a equals 90° — $\propto$, hence Ob also varies with $\propto$. Now the distance ab, when Oa equals 1 and the angle at O is $\propto$, is called Sine $\propto$ and the distance Ob is called Cosine $\propto$.

In the triangle lmn, similar to Oab

mn : ab (Sin. $\propto$) = ml : Oa (1) $\therefore$

$$mn = ml \text{ Sin.} \propto \therefore \text{Sin.} \propto = \frac{mn}{ml}$$

In the same way it may be proved that Cos. $\propto = \frac{nl}{ml}$

These may be expressed thus,

$$\text{Sin.} \propto = \frac{\text{side opp.}}{\text{hypot.}}$$

$$\text{Cos.} \propto = \frac{\text{side adj.}}{\text{hypot.}}$$

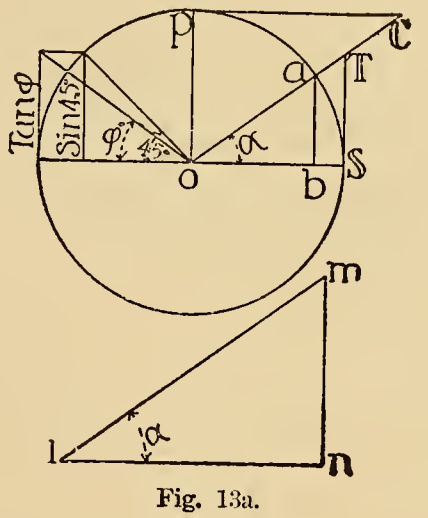

Fig. 13a.

These are very useful equations, for if one angle and one side of any right angled triangle are known, all the other parts are easily found by the use of a table of Trigonometrical Functions.

If we take OS equal to Oa and erect a perpendicular at S cutting Oa produced in T, we have in TS what is called Tangent $\propto$ and, by an evident construction, we have in pC what is called Cotangent $\propto$.

Now TS : ab = OS : Ob; hence, as OS equals 1, TS = $\frac{ab}{Ob}$ or Tan. $\propto = \frac{\text{Sin.} \propto}{\text{Cos.} \propto}$.

In the triangle pC, similar to Oab, angle C equals $\propto$ and $\frac{Op}{pC}$ = Tan. C = Tan. $\propto$; but Op = 1 and pC is Cot. $\propto$ and Tan. $\propto = \frac{\text{Sin.} \propto}{\text{Cos.} \propto}$; hence $\frac{1}{\text{Cot.} \propto} = \frac{\text{Sin.} \propto}{\text{Cos.} \propto}$ and Cot. $\propto = \frac{\text{Cos.} \propto}{\text{Sin.} \propto}$.

In connection with logarithmic tables will be found tables of Logarithms of Trigonometrical Functions and also tables of their natural values. Thus it will be found that Sin. 45° = Cos. 45° = .7071. Now as Tan. Φ = Sin. 45° (Fig. 13a, left side) the angle Φ can be found in the tables by looking *for the angle whose tangent is* .7071; *it is approximately* 35° 16′.

CHAPTER II.

SHADOWS ON THE VERTICAL PLANE OF PROJECTION.

APPLICATIONS.

§ 5. Shadow Cast on the Vertical Plane of Projection by a Point and by a Straight Line in Different Positions.

(a) Shadow of a point. Let aA (Fig. 15) be a point at a distance δ from the vertical plane.

We have in a_1A_1 its shadow on the wall. It is found by determining the point of intersection of the vertical plane with the luminous ray through aA.

It will be noticed that the shadow A_1 in elevation is in a 45° line from A (elevation of the point) at a distance equal to the diagonal of a square whose side is the distance of the point in front of the wall.

Hence, if this distance is known, the shadow may be found without recourse to the plan.

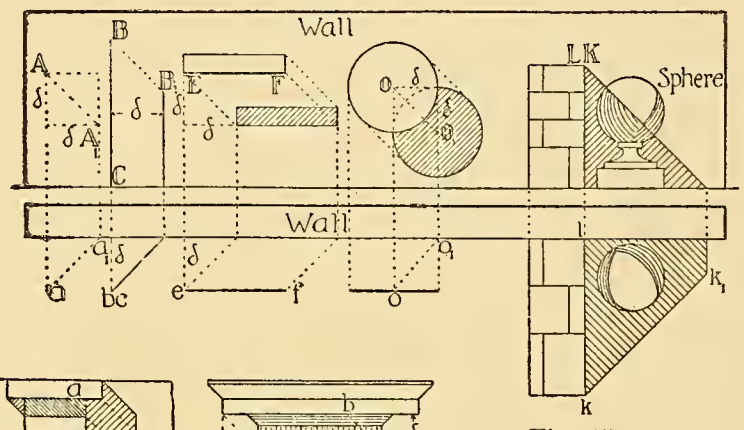

Fig. 15.

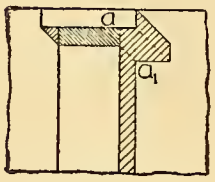

Fig. 16.

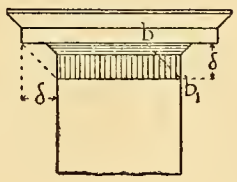

Fig. 17.

(b) Of a vertical line BC and of a horizontal line EF, parallel with the vertical plane. BC casts shadow in B_1C_1 at a distance to the right equal to the distance of the line from the wall. Likewise EF casts a shadow in a line below it at a distance equal to the distance of the line from the wall.

(c) Of a plane figure parallel with the wall. (A circle for example.)

This figure casts a shadow its own size and form. The shadow is found by displacing the elevation to the right and downward by a distance equal to the diagonal of a square whose side is the distance of the figure from the wall. For a circle it is sufficient to find the shadow of the centre by means of its distance from the wall, and to draw the shadow equal to the given circle.

(d) Of a line in profile. (Fig. 15.)

Take for example the edge of a wall, perpendicular to the vertical plane. The plane of shade cast by such a line is itself perpendicular to the vertical plane, and, consequently, the shadow, which is the intersection of this plane with the wall (or with any other object, a sphere for example), is a line at 45°.

(e) Of a line in profile plane inclined to the horizontal at an inclination $\frac{m}{n}$ (m rise to n run).

In finding the shadow AB_1 (Fig. 18), it is evident that this line is inclined to the horizontal at the inclination $\frac{m+n}{n}$ (m+n rise to n run).

Example.—A line in profile whose inclination is $\frac{0}{1}$; then the inclination of its shadow equals $\frac{1}{1}$, that is to say it is inclined at 45°, as we have already found in Fig. 15.

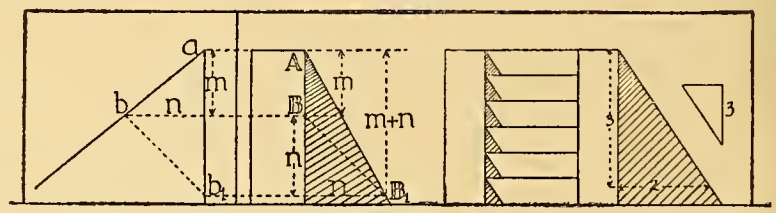

Fig. 18.

A line in profile plane at 45°, that is, with inclination $\frac{1}{1}$, has its shadow inclined at $\frac{1+1}{1} = \frac{2}{1}$.

A stair ramp (Fig. 18) has generally an inclination of $\frac{1}{2}$; the inclination of its shadow is $\frac{2+1}{2} = \frac{3}{2}$. (This last result should be committed to memory.)

§ 6. SHADOWS OF A CIRCLE AND OF A CUBE ON THE VERTICAL PLANE OF PROJECTION.

A circle ordinarily occurs parallel with the wall, horizontal or in vertical profile plane.

Suppose we have circles in three faces of a cube whose elevation is ABCD (Fig. 19). First find the shadow of the cube. Let ABCD be the face nearest the wall, and MNPQ the face furthest from the wall.

The distance of the nearest face from the wall is supposed to be known. By means of this we find the shadow of that face, as shown in $A_1B_1C_1D_1$, by laying off the distance S to the right and then downward; the shadow is a square.

The square MNPQ is further from the wall than the other by a distance equal to the side of the cube; its shadow is in $M_1N_1P_1Q_1$, found by displacing the first square a distance equal to its side, downward and again to the right.

This makes the diagonal M_1P_1 a prolongation of A_1C_1 and makes the two points M_1 and C_1 coincide. Then join the angles A_1M_1 –B_1N_1 –D_1Q_1 –C_1P_1 by 45° lines, and the figure is on an oblique projection of the cube, or its shadow.

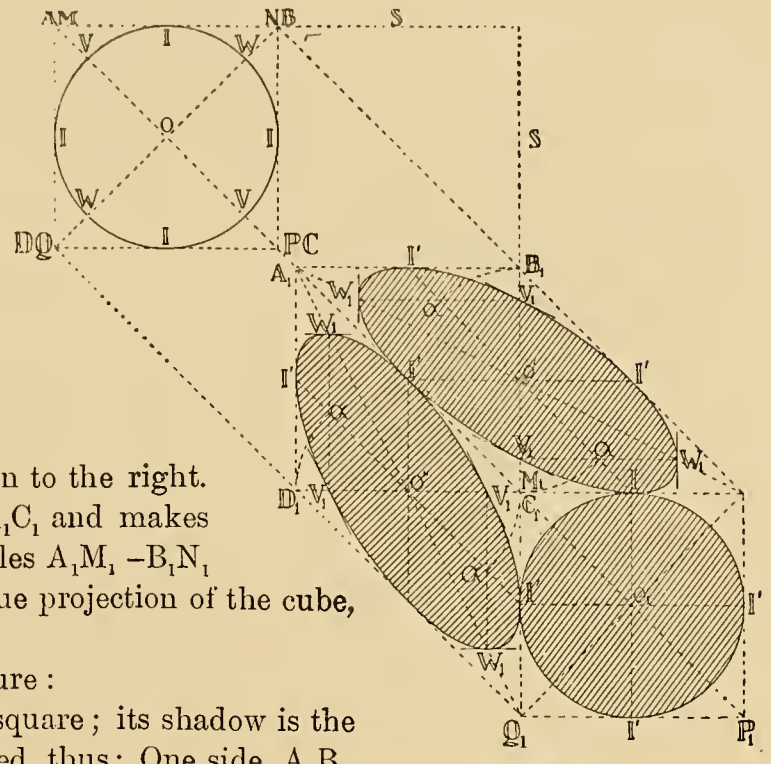

Fig. 19.

We may note the following in reference to this figure:

FIRST. The upper face of the cube is a horizontal square; its shadow is the parallelogram $A_1B_1M_1N_1$, whose form is easily remembered, thus: One side, A_1B_1, is horizontal, the other is at 45°; its short diagonal B_1M_1 is vertical and equal to the shortest side. The other diagonal A_1N_1 is inclined at $\frac{1}{2}$. Two 45° triangles placed side by side give the form of this parallelogram.

SECOND. If we inscribe a circle in the upper face of the cube the circle will cast shadow in an ellipse inscribed in this parallelogram. This ellipse will be tangent to the sides of the parallelogram at their middle points, I′I′, etc.

The line $O'B_1$ and $O'A_1$ will be cut by the ellipse into the same proportional lengths as are OB and OA. But $O'I' = OB$ and $O'B_1 = OW$. Thus $\propto$ divides $O'I'$ just as W divides OB. A parallel through $\propto$ will divide $O'B_1$ and $O'A_1$ in the same proportion at V and W.

The points V_1V_1 will be called the "points of the short diagonal" and W_1W_1 "points of the long diagonal."

Note that the tangent to the circle at V is parallel to OB and the tangent at W is parallel to OA. In the shadow the same parallelism is preserved by the tangents to the curve at V_1 and W_1. Hence the tangent at V_1 inclines at $\frac{1}{2}$ and the tangent at W_1 is vertical.

Such is the ellipse of shadow cast by a horizontal circle. (The student should learn to draw it without hesitation and without finding unnecessary points.)

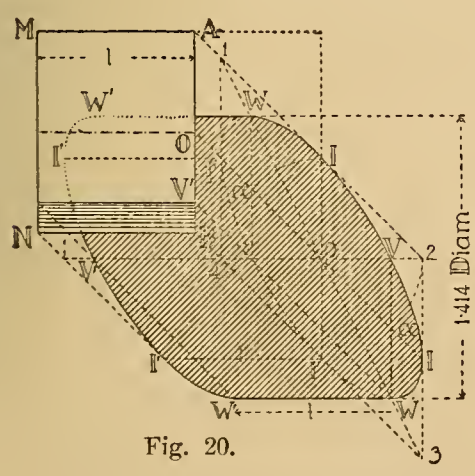

Fig. 20.

THIRD. The square in profile casts its shadow in the parallelogram $A_1D_1Q_1M_1$ and the circle in profile plane casts the ellipse inscribed within it. This is the same ellipse as the preceding, except that it is reversed.

The important points of this ellipse are the middle points I′I′, the points of the short diagonal V_1,V_1, and the points of the long diagonal W_1W_1. At points V_1V_1 the tangent is inclined at $\frac{2}{1}$; at point W_1W_1 it is horizontal.

§ 7. APPLICATION TO ARCHITECTURAL FORMS.

(a) Shadow of a drum of a column whose axis is horizontal and parallel with the vertical plane. (Fig. 20.)

We find in oV′ the shadow of the axis, knowing the distance of the axis from the wall. Construct on O as centre the shadow of a square in profile plane, and inscribe the ellipse as shown in the previous

paragraph. The tangents at the points WW of the long diameter are horizontal, and give the shadows cast by the lines of shade of the cylinder.

By passing back along a ray from the point of tangency W to V′ on the cylinder we have a point of the shade on the cylinder. The separatrix is the generatrix passing through this point.

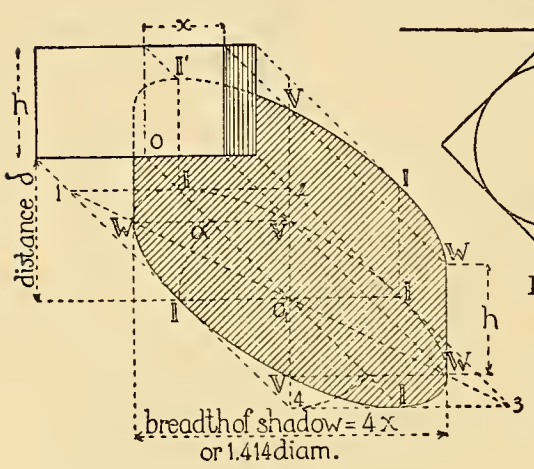

Fig. 21.

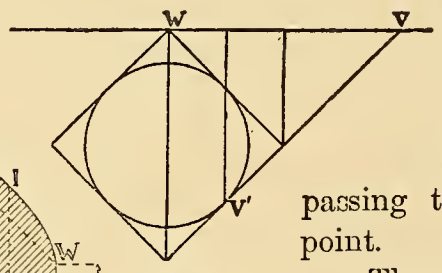

Fig. 21a.

The shadow of the left base is found by displacing the first ellipse a distance equal to the length of the cylinder.

REMARK.—*The breadth of the shadow of the cylinder is greater than the diameter d of the cylinder. It equals four times the distance OV′, between the line of shade of the cylinder and the projection of the axis.*[1]

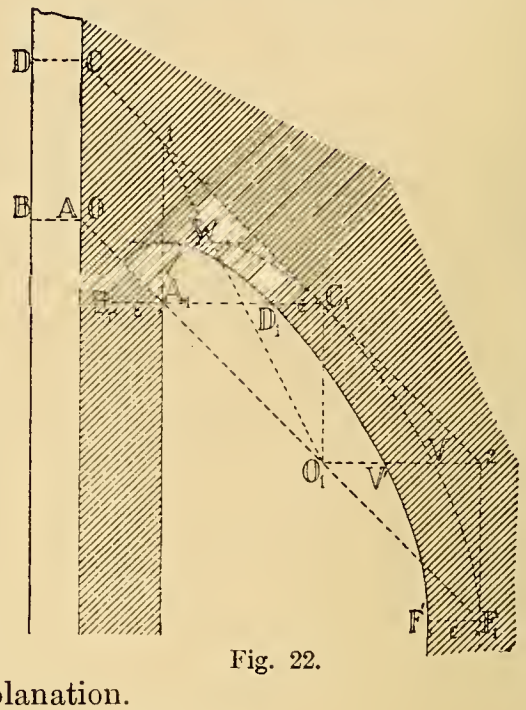

Fig. 22.

(b) Shadow of a drum of a column. Axis vertical. (Fig. 21.)

The process is analogous to that of the preceding, and needs no explanation.

(c) Shadow of an arch in profile and of the same arch in section. (Figs. 22 and 23.)

Following the same method as before, draw the shadow of a semicircle in profile plane two times.

If the arch is cut through its axis by a plane parallel with the vertical plane of projection, the shadow will be composed of one-half the semi-ellipses preceding, joined by the line D_1C_1 shadow of DC, the line of section of the key of the arch. On the intrados at VK, we have a fragment of the ellipse B_1K_1, intercepted by the intrados. (Construction will be given later under title of "Bridge Shadow.")

Note K_1, the point of intersection of the ellipses.

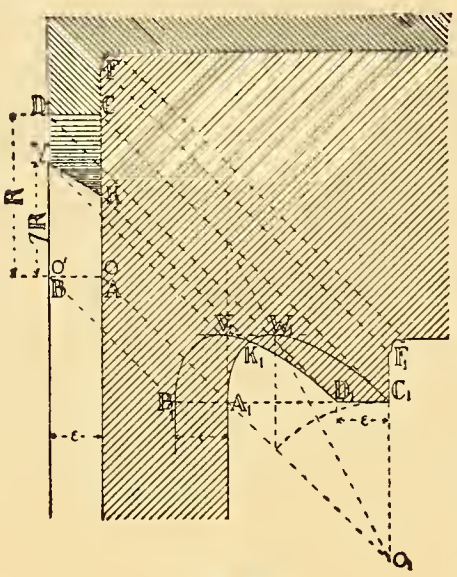

Fig. 23.

(d) Shadow of a cylindrical opening in the ceiling of a square hall.

Fig. 24 needs no explanation. It is analogous to Fig. 23, except that in place of a vertical semicircle we have to consider the shadow of a horizontal semicircle.

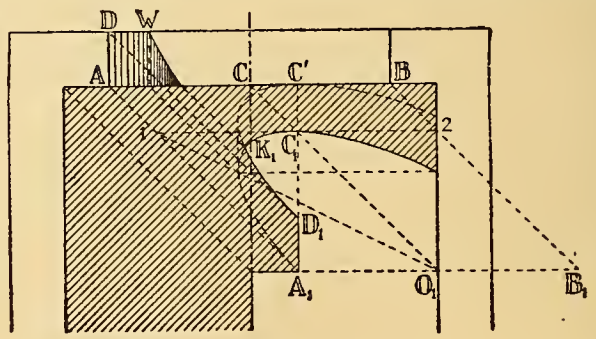

Fig. 24.

[1] In Fig. 21a the plan of the cylinder of Fig. 21 is inscribed in a square, one corner of which is in the vertical plane of projection. Pass rays of light tangent to the cylinder; then the breadth of the shadow on the vertical plane will be WV and V′ will be a point of the line of shade on the cylinder. Then it is evident from the figure that WV equals four times OV′.

It will be noted, however, that WV is the diagonal of a square whose side is the diameter of the cylinder. A 45° line may be drawn across the elevation of the cylinder, and the diagonal thus found taken for the breadth of the shadow.

CHAPTER III.

SHADOWS OF INTERRUPTED PROFILE.

§ 8. Shadows of a Corona on a Series of Planes—All of Which are Parallel with the Vertical Plane of Projection.

Let there be a line MN (Fig. 25), parallel with the ground line, forming the edge of the corona, and casting shadow on a series of planes parallel with the vertical plane of projection.

It is easily seen that the shadow offsets in the line ab–b_1c–c_1d, etc., and that it shows the successive projections of the wall,—that is to say the profile of the shadow, if unbroken by the shadows of the vertical lines, would be the same as the plan of the wall.

The shadow of c, lying in the edge, is c_1 in the primitive shadow from the corona MN. But this is also a point of the shadow cast by the edge Cc, and determines the location of the vertical line of shadow.

A point such as c_1 is called a "point of change." In this point the shadows of two different lines cross.

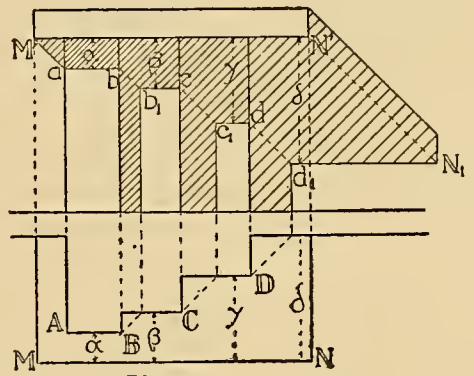

Fig. 25.

§ 9. Architectural Application.

(A) Shadows Cast by a Vertical Line Upon Moldings Parallel with the Ground Line. (Fig. 26.)

First. The profile of the shadow. The line of shadow reproduces the actual profile of the moldings.

It is thus possible to draw the line of shadow, without making direct use of the section, if the profile is known and the distance of any part of the molding from the line.

The section (Fig. 26) gives the profile of the molding. Knowing the distance of the line at 1, back of the vertical line NR, which casts the shadow, we may draw, beginning at a, the profile of the molding, copying from the section. (Hence it is needless to draw the section.)

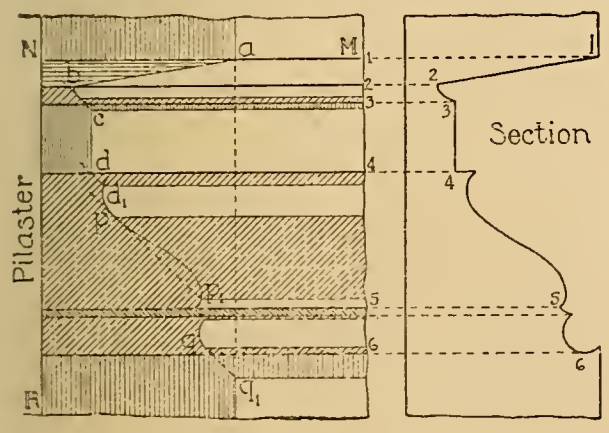

Fig. 26.

This is the geometrical shadow cast by the line NR on the molding. It is called the "unbroken profile" of the shadow. It is, however, interrupted by the shades on the curved portions of the molding and by the shadows cast by one portion on another.

Second. The interruption of the shadow profile. The shadow is completed as follows: (a) The shadow of NR drops from d to d_1; through d_1 pass the horizontal shadow cast by the molding upon itself.

(b) Draw lines at 45° tangent to the curves at points of loss, such as p and q, and these are found to be at the same time points of the horizontal lines of shade.

(c) These tangents, prolonged until they cross the "unbroken profile," give, in p_1 and q_1, points of other horizontal shadows cast by the molding on itself.

(B) Shadow on Vertical Moldings of a Horizontal Line which is Parallel to the Vertical Plane of Projection. (Fig. 27.)

It is the shadow cast by a line, parallel to the ground line M′N′, on a series of vertical moldings. The

profile of the molding is reproduced, as in the preceding figure, in abcd etc. This is the unbroken profile of the shadow.

We then distinguish the limiting points, b and d, and the points of loss, p and q; we find the points of change as before after having determined the interruptions to the shadow at p and q. These points of change, p_1 and q_1, serve as points of departure for the vertical shadows cast by other parts of the molding.

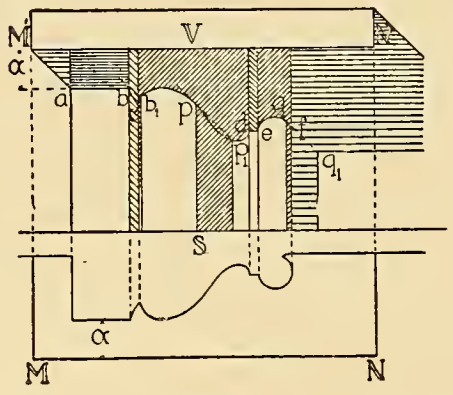

Fig. 27.

§ 10. Shadow Cast by a Vertical Line on an Inclined Plane which is Parallel to the Ground Line. (Fig. 28.)

This is a particular case of the preceding problem. The shadow is a line $c'a_1$, inclined to the horizontal under the same angle as the plane makes with the horizontal plane, and beginning at c′, at the distance b′c′ from the foot of the line, equal to the distance of the line in front of the edge of the given plane.

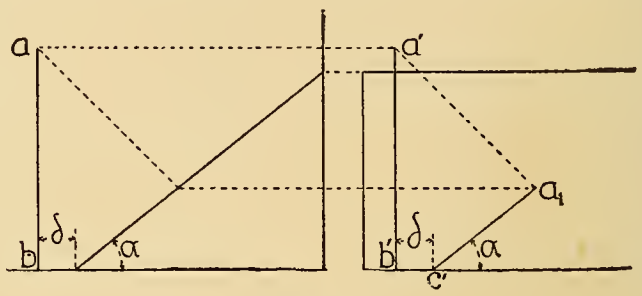

Fig. 28.

§ 11. Architectural Applications.

(a) Shadows on a stairway with sloping abutment. The stair has the ordinary inclination, one rise to two run. In elevation (Fig. 29) the portion of the abutment in profile plane gives a line of shadow at 45° from A. The projection ∝ determines the position of A_1, and the shadow A_1L_1, cast by the inclined portion of the abutment, is inclined at $\frac{3}{2}$. (See § 5.) Likewise the small shadows cast on the risers of the steps are inclined at $\frac{3}{2}$. The upper ends of these short lines, on the edges of the steps, are all in the same vertical line, whose distance, e, to the right of AB is $\frac{2}{3}$ the height d, of the line AB above the imaginary plane gh (Figs. 31 and 32), which passes through all the salient edges of the steps. Fig. 32, at a larger scale, shows why this distance e is $\frac{2}{3}$ of the height fg.

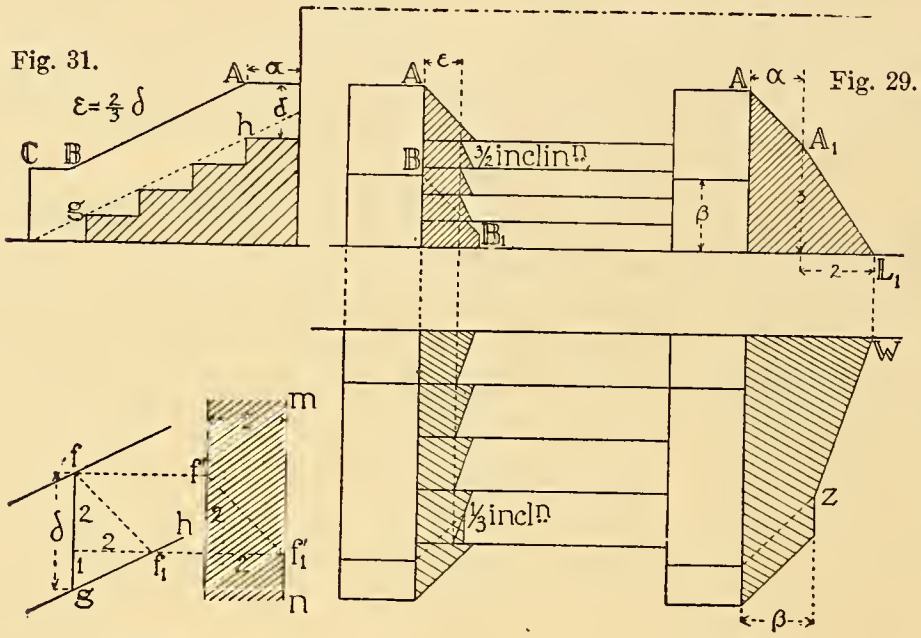

Fig. 32. Fig. 30.

In plan the abutments cast shadows, on the ground and on the treads, which are inclined at $\frac{1}{3}$ $\left(\frac{m}{m+n}\right)$, etc.

(b) *Shadows of a pedestal on a stair.* (Fig. 33.)

First. The projection (supposed to be known) of the corner gh in front of the riser No. 2 gives at 2 the shadow on the riser of step No. 2.

Second. Through the point 2 draw the line 1 2 3 at the same inclination as that of the stair (one rise to two run), and by means of this complete the profile of the shadow.

Third. Find the shadow m_1g on the pedestal, and g_1, by a line through the limiting point g, etc.

The remainder is easily understood from the diagram.

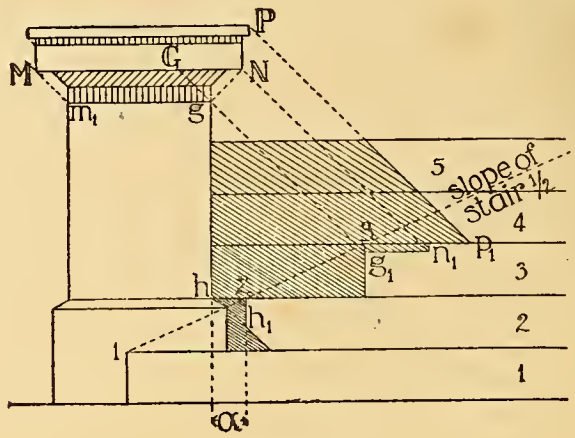

Fig. 33.

(c) Shadow of a chimney on a roof. (Fig. 34.)

FIRST. From Z, the foot of the edge of the chimney, draw ZW at the same inclination as that of the roof (supposed to be known).

SECOND. Laying off from Z the depth of the chimney (known) at 2 we find on the line ZW the point 1 and afterward 1′, the foot of the back left corner of the chimney; 1′3, parallel with ZW, is the shadow of this edge.

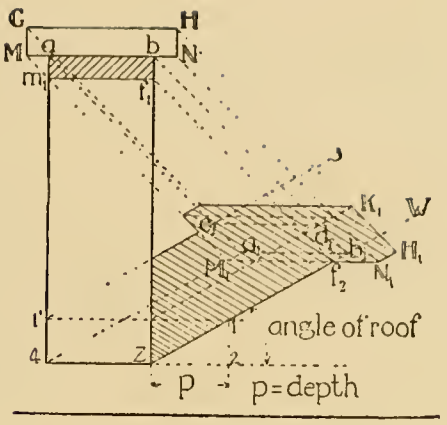

Fig. 34.

THIRD. In $a_1c_1d_1b_1$ we have the shadow cast by the top of the shaft (at ab).

FOURTH. m_1f_1 is the shadow of the cap on the shaft of the chimney.

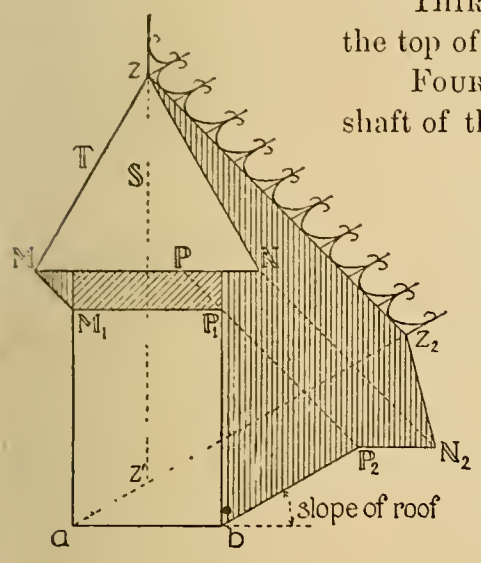

Fig. 35.

FIFTH. The shadow of the cap on the roof is at $M_1N_1H_1K_1$, and it will be noted that a_1 and M_1 are both on the line $4a_1M_1$ inclined at the angle of the roof and drawn from the point 4; likewise K_1 and d_1 are both on a line having the same inclination as the roof.

(d) Shadow of a dormer on a roof. (Fig. 35.)

The vertical bP_1 gives a shadow bP_2 inclined at the angle of the roof, beginning from b where the corner bP_1 pierces the roof, etc.

The shadow of the cresting, a mere matter of detail (see the figure), is drawn more or less from the imagination; we have only to incline the shadows of the vertical members of the cresting at the same angle as the roof.

§ 12. SHADOW OF A HORIZONTAL CIRCLE ON A VERTICAL WALL RECEDING AT 45°.

The plane Q (shown in perspective, Fig. 36), which is vertical and forms the angle 45° with the vertical plane of projection, is cut by a horizontal circle; suppose that the centre O is in the plane Q. We shall call such a plane an *oblique plane at 45°*.

Projection drawing. (Fig. 37.) Two diameters of the circle, VW and fd, at right angles, give two conjugate diameters of the ellipse of shadow, f_1d_1 and V′W′. It will be noted, 1st. That these latter diameters are at right angles; 2d. That they are equal; hence *the shadow takes the form of a circle.* This circle passes through the 45° points V and W.

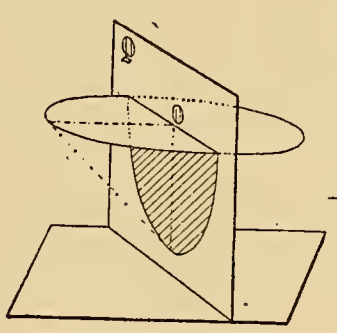

Fig. 36.

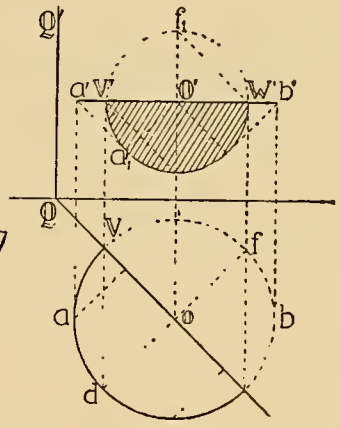

Fig. 37.

This proposition is particularly useful in the problems which are to follow, and greatly facilitates the application of the method of oblique projections.[1]

[1] It is quite essential that the reader make all these drawings; make them at a larger scale, and take other figures than those given here. He will not find, then, exactly the same results, and in observing the differences the study will become much more profitable to him. A thorough mastery of Shades and Shadows cannot be gained by merely reading the book without making the practical applications.

CHAPTER IV.

LINES OF SHADE ON SURFACES OF REVOLUTION.

§ 13. SHADE ON A CONE. (Fig. 38.)

General Method.—1st. Take any auxiliary plane P. Find upon it the shadow a_1b_1 of the base and the shadow S_1 of the apex of the cone S; 2d. Draw the two tangents S_1m_1 and S_1n_1; 3d. Pass back from m_1 to m and from n_1 to l on the base; the shade is Sm and Sl.

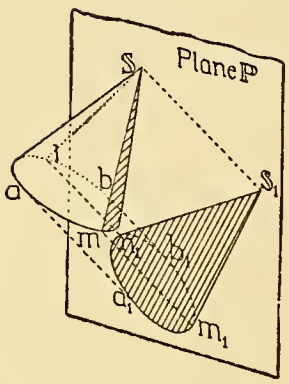

Fig. 38.

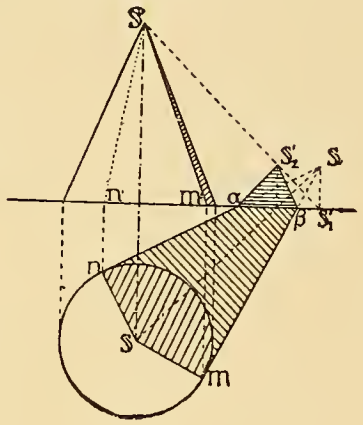

Fig. 39.

REMARK.—The plane P may be chosen to pass through either the base ab (when there is no need of finding the shadow of that base) or through the apex (when there is no need of finding the shadow of the apex).

(a) Projection drawing. *First Method.*—By finding the shadow on a horizontal plane. (Fig. 39.)

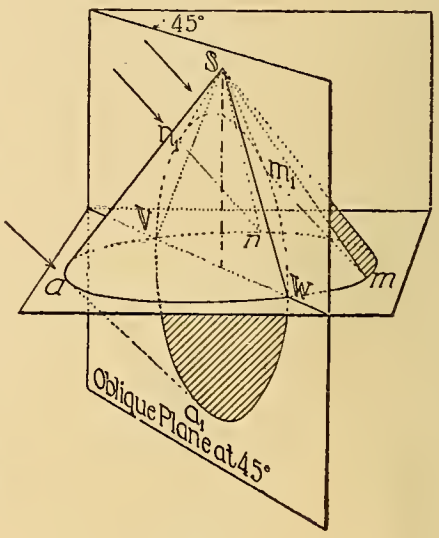

Fig. 40.

1st. S_1 is the shadow cast by the apex on the horizontal plane of the base; 2d. Draw the two tangents S_1m and S_1n to the base (which is its own shadow), this gives the shadow of the cone; 3d. The lines of shade are Sm—S′m′ and Sn—S′n′; 4th. Find at S'_2 the shadow of the apex on the wall, and join it with the "points of folding" ∝ and β, where the first shadow cuts the ground line; this gives in ∝ S'_2β the shadow cast on the wall.

(b) *Second Method.*—Using the oblique plane at 45° (Fig. 40, shown in perspective). 1st. Assume the oblique plane at 45° passing through the axis and consequently containing the apex S; 2d. Find in Va_1W the shadow of the base (§ 12). We know that it passes through the 45° points of the base, V and W, and that in projection this shadow will appear as a circle; 3d. The apex S, being in the plane, is its own shadow. 4th. Draw the tangents Sm_1 and Sn_1 to the shadow of the base; then the figure $Sn_1Va_1Wm_1$ is the shadow of the cone on plane P; 5th. Pass rays through m_1 to m and through n_1 to n on the base; then the shade of the cone is Sm and Sn.

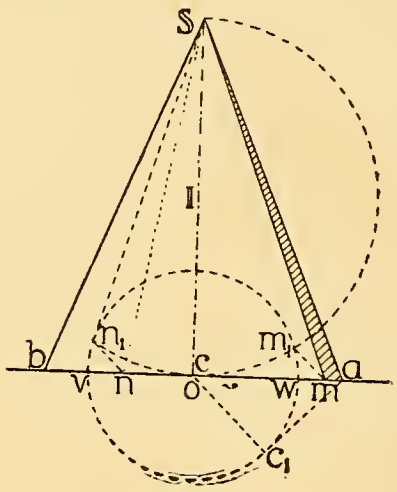

Fig. 41.

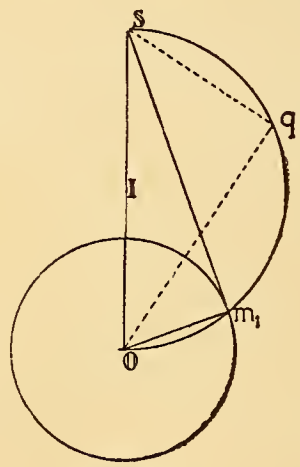

Fig. 41a.

Projection drawing. (Fig. 41.) Let Sab be the elevation of the cone, the axis being vertical.

1st. Find the circle Vc_1W, shadow of the base on the oblique plane at 45° (see § 12); 2d. Draw tangents to this from the point S, and find in m_1 and n_1 the points of contact; these points of tangency may be found with greater exactness by describing a circle n_1m_1S with the centre at I, which is the middle of SO[1]; 3d. Passing 45° rays through m_1 and n_1 we find m and n on the base; the lines of shade are Sm and Sn.

[1] A line is tangent to a given circle at a given point when a radius forms a right angle with the line at that point. The familiar geometrical proposition that any two contiguous lines which, together, subtend a semicircle (as Sq and qO or Sm_1 and m_1O, Fig. 41a) are at right angles, enables us to find the exact point of tangency by the arc of a circle. The line m_1O being understood, need not be drawn.

§ 14. 45° CONE—CONE OF ANGLE φ—CONE WITHOUT SHADE.

FIRST. Cone whose angle at the base is 45°. (Fig. 42.) Using the construction of the last paragraph we find that one of the lines of shade, Sa, is in the contour Sa—S′a′ of the cone; the other, Sc, is a vertical line in the elevation. The cone has then only a quarter of its surface in shade, and this shade is entirely hidden in elevation.

SECOND. Cone whose angle at the base is φ. (Fig. 43.) A ray through the apex S–S′ does not leave the surface of the cone, since the cone is itself inclined at the angle φ. Now S_1, shadow of the apex, falls on the base and the shade is reduced to a line S_1S. In reality there is no shade.

THIRD. Hence it is evident that any cone having at the base a less angle than φ has no shade.

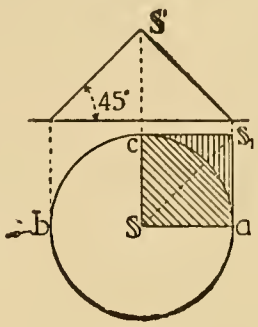

Fig. 42.

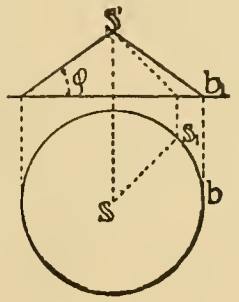

Fig. 43.

§ 15. SHADE ON A CYLINDER. (Fig. 44.)

Considering a cylinder as a cone whose apex is at infinity, we may find the shadow by drawing two tangents at 45°. The lines of shade are then verticals through m′ and n′. They pass through the 45° points of the base, n and m, which are easily found by the construction shown at O′ca. (Fig. 44.)

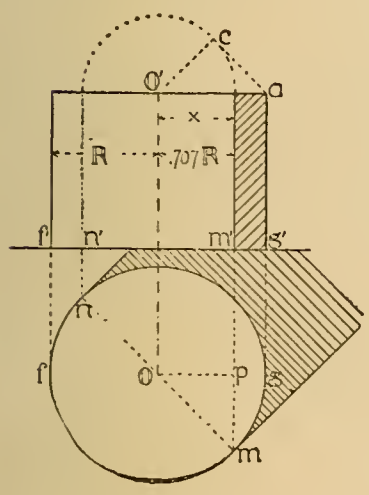

Fig. 44.

The distance x equals Op, which is one side of the 45° triangle Opm. By trigonometry Op is .707 of Om, or x is about .7 the radius.

The triangle O′ca is equal to Opm; O′c = Op, and is applied to the cylinder by the arc.

§ 16. SHADE AND SHADOW OF THE SPHERE. (Fig. 45.)

1ST CASE. For simplification the ray of light is made parallel with the vertical plane of projection and thus makes the angle φ with the ground line.

(a) *Shade.*—The ray of light being parallel with the vertical plane, the circle of contact of the cylinder of shade with the sphere (the plane of which is perpendicular to the ray of light and also to the vertical plane), is shown in the elevation by the line m′n′, perpendicular to the ray φ.

In plan this will be an ellipse, which is the projection of the circle m′n′; its two axes are cd and mn.

Calculation of y, half the minor axis.—In elevation we have o′u′ equal to y. o′u′m′ is a right angled triangle whose angle at m′ is $\varphi = 35° 16'$. By trigonometry the Sine of $\varphi = .577$ approximately, and $y = .577$ o′m′; if o′m′ be called R, $y = .577$ R. But R is $\frac{1}{2}$ the major axis of the ellipse, hence the major axis is to the minor axis as 1 is to .577, or very nearly as 7 is to 4.

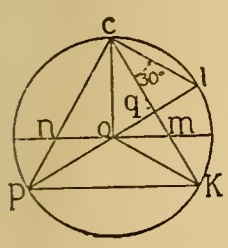

Fig. 45a.

But om equals $\frac{1}{3}$ of the side of an equilateral triangle inscribed in the contour.[1]

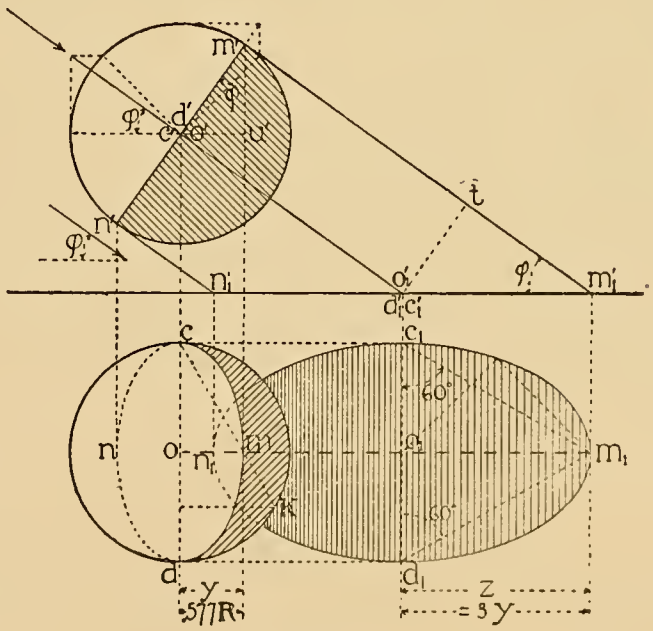

Fig. 45.

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[1] (Fig. 45a)—The line cl = R (side of inscribed hexagon) and the triangle qcl is a right triangle with 30° at c; by trigonometry cq = .866 of cl and ql = .5 of cl; hence ql = $\frac{1}{2}$ R and oq = $\frac{1}{3}$ pq.

The triangles omq and pkq are similar, hence om is $\frac{1}{3}$ pk. As further proof we may note that in the triangle oqm we have oq (= $\frac{1}{2}$ R) = .866 om, hence om = $\frac{\frac{1}{2}R}{.866}$ = .577 R. Q. E. D.

Practical construction for the ellipse of shade:

Take dK = R; join Kc, and the intersection m with the central ray oo_1 gives the extremity of the minor axis. Complete the ellipse.

(b) Shadow cast on the horizontal plane. (Fig. 45.) It is the ellipse $n_1m_1c_1d_1$. Its minor axis c_1d_1 equals the diameter of the sphere and is perpendicular to the ray of light. Its major axis z equals $o'_1m'_1$. Consider the right triangle $o'_1tm'_1$ with angle φ at m'_1. Then by trigonometry $R = .577\ o'_1m'_1$ $(= z)$ and $z = \frac{R}{.577}$ or $z : R = 1 : .577$. But in the last paragraph we found that $R : y = 1 : .577$, hence $R : y = z : R$ and the ellipses of shade and shadow have the same proportions of length to breadth.

Now $y = .577\ R$ and $z = \frac{1}{.577} R$, whence it will be found that $z = 3y$. It will be further noticed that cmn is an equilateral triangle.

Construction of the ellipse of shadow: Take z equal to 3 times y; or on cd as base construct equilateral triangles; the summits m_1 and n_1 of these triangles are the extremities of the major axis. The minor axis equals the diameter of the sphere.

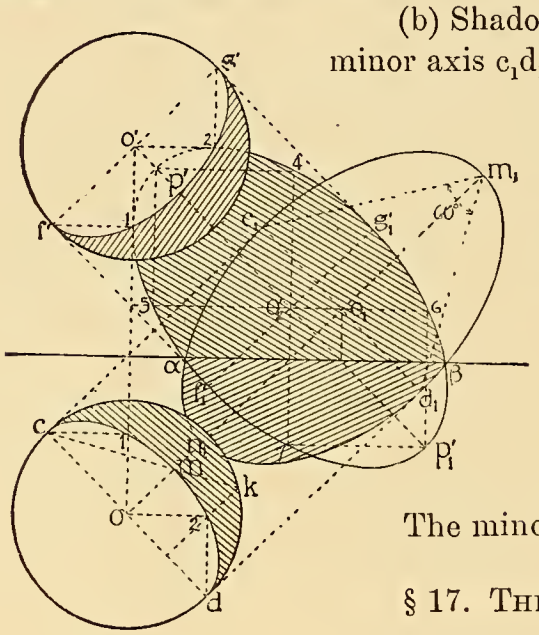

Fig. 46.

§ 17. THE USUAL CASE OF THE SHADE AND SHADOW OF THE SPHERE. (Fig. 46.)

Apply the results obtained in the last paragraph, only setting them at 45°.

FIRST. In plan we have in O_1 the shadow of the centre. The minor axis of the ellipse of shadow is c_1d_1 perpendicular to the ray of light and equal to the diameter of the sphere; the major axis n_1m_1 is obtained by equilateral triangles.

The major axis of the shade, cd, equals the diameter of the sphere; its semi-minor axis is om, found as shown in the last paragraph.

NOTE.—The 45° points c and d, projected on the diameters O1 and O2, give points of the ellipse at 1 and 2 (becomes evident on comparison of plan and elevation).

SECOND. Shadow on the vertical plane. Having in O'_2 the shadow of the centre on the wall (determined by the distance of the centre from the wall), complete the ellipse exactly as in the plan, by using the equilateral triangles.

REMARK.—The "points of folding," α and β, where the horizontal ellipse of shadow cuts the ground line, are also on the ellipse cast on the wall.

§ 18. SHADE ON ANY SURFACE OF REVOLUTION.

(a) Shade on a torus. Elevation. (Fig. 47.)

The points on the contour m and n are found by tangents under 45° (see § 3, Theorem of Contours). On account of symmetry with reference to the 45° meridian, the point m″ on the axis is found by drawing a horizontal mm″. The 45° point on the equator, k, is found by the arc oSk (the same as for the shade on the cylinder, application of the method of circumscribing surfaces). In order to find the lowest point, c′, apply the method of cutting planes. Suppose that the 45° meridian plane, parallel with the light, be revolved backward until

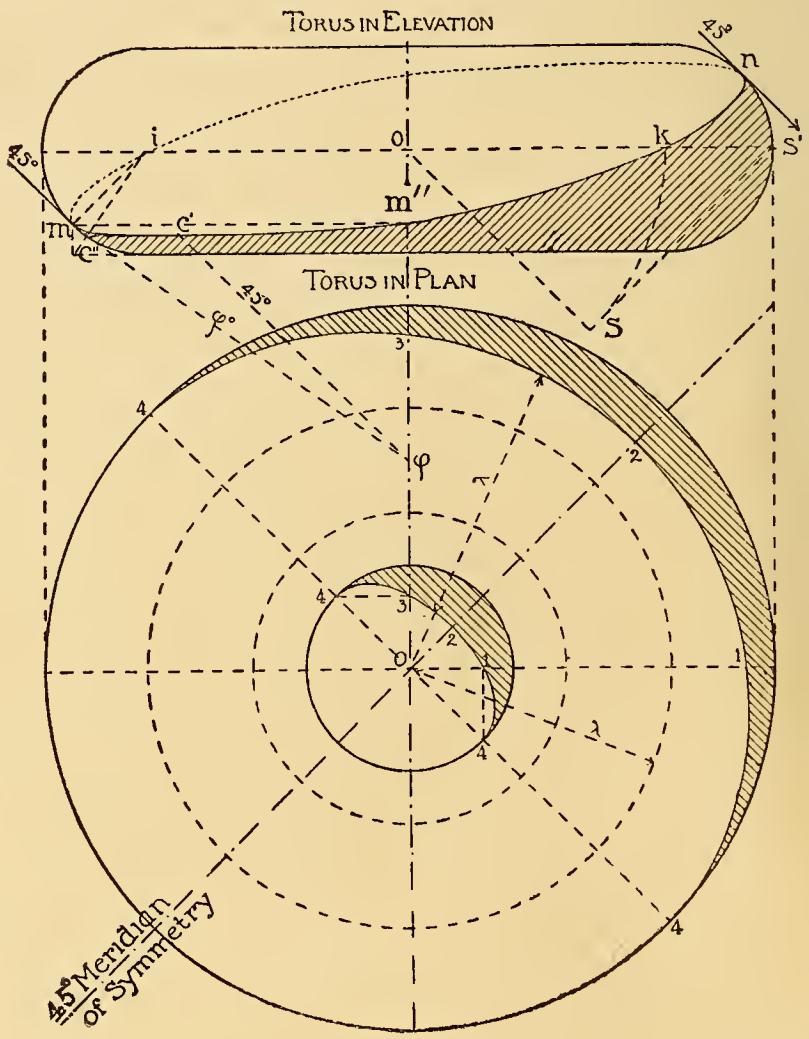

Fig. 47.

it becomes parallel with the vertical plane of projection, carrying with it the ray of light, which then appears at the angle φ.

Then draw $\varphi c''$ tangent to the principal meridian under the angle φ; revolve c'', the point of tangency, back to its place at c'. In this revolution the point φ on the axis remains fixed, and the point c'' is displaced horizontally. The ray at φ becomes the ray at 45° and the point c'' will be found at c' on the 45° line through φ.

In this way 5 points are readily found and also the tangents at three of these points, n, m, c'; these will generally be found to be sufficient in practice.

It is evident that the same principles will apply in finding the shade on other surfaces of revolution.

(b) Shade on a torus. In plan. (Fig. 47.)

1st. Points on the equator, 4, 4, are found by 45° tangents (Theorem of Contours).

2d. Points 1, 3, on the two co-ordinate meridians may be found by drawing 45° tangents to the principal meridian in the elevation, and transferring the points of tangency to the plan.

The highest point, 2, in the 45° meridian of symmetry may be found by drawing a tangent to the contour (in elevation) under the angle φ, and then transferring the point of tangency to its proper place in the plan.

REMARK.—The shade in the plan is what is called a concoid of the ellipse 4 2 4, which is the shade of the central sphere (a sphere of same radius as the section of the torus).

This is obtained geometrically by drawing such radii as $O\lambda$, taking their intersection with the shade of the central sphere, and laying off a distance λ, constant and equal to the distance from the centre of the torus to the centre of the meridian circle.

CHAPTER V.

SHADOWS OF CONVEX CYLINDERS.

§ 19. Shadow of an Abacus. (Fig. 48.)

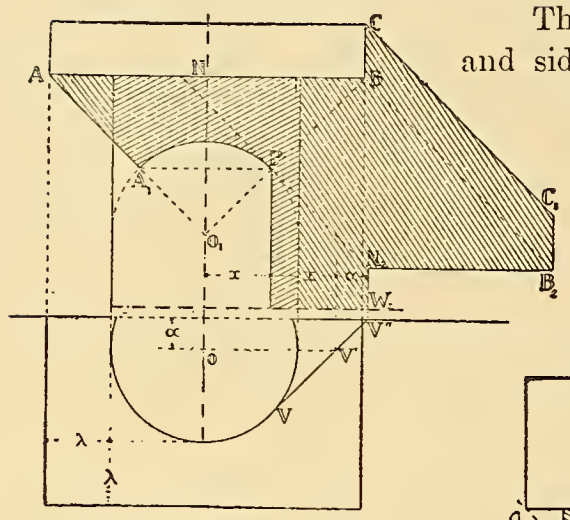

Fig. 48.

The abacus has a projection λ beyond the shaft of the column at front and sides. The axis of the column stands in front of the wall at a distance $\propto$, ordinarily equal to $\frac{1}{3}$ R (R being the radius of the shaft).

The shadow on the shaft is, according to Chapter III, a circle described about O_1 as a centre. This circle is only used between A_1, shadow of the left angle A of the abacus, and P, the point of loss.

The vertical line through P is the shade on the shaft; N_2 is a point of the shadow on the wall and its distance from the shade line is $x + \propto$.

The rest of the drawing is easily comprehended; note that no plan is needed; the knowledge of the distance $\propto$ is sufficient.

§ 20. Application.

Doric guttæ and their fillet. (Fig. 49.) The profile at the left enables us to find in a_1c_1 the shadow of the architrave fillet on the fillet of the guttæ and in a_2c_2 the shadow of the latter on the architrave (interrupted by the guttæ).

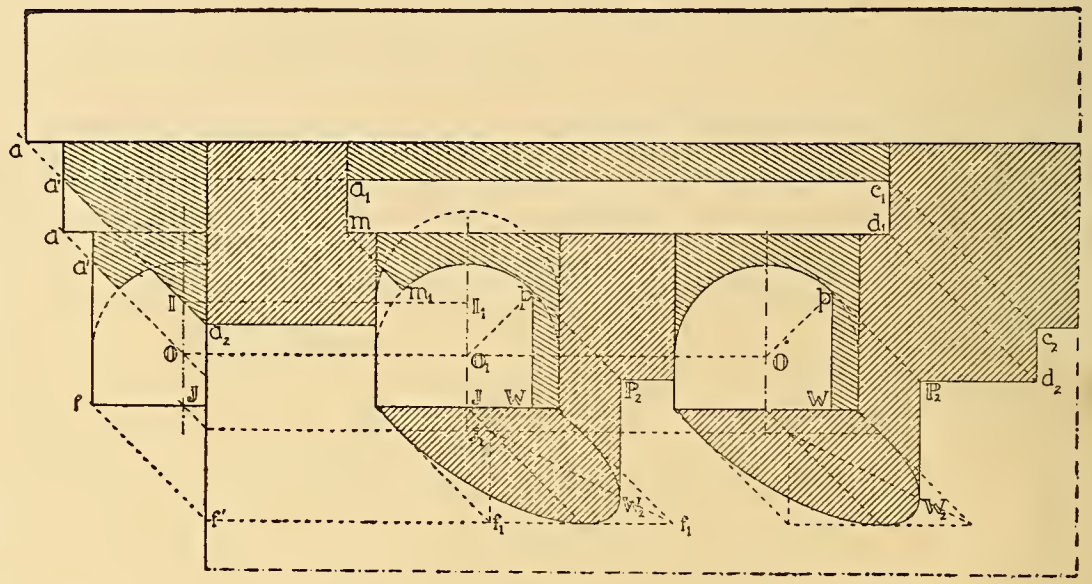

Fig. 49.

Suppose that the guttæ are cylinders whose axes are at distance $\propto$ from the wall. The fillet md_1 casts shadow on the guttæ in arcs of circles having O_1 and O as centres. The points of loss, p,p, correspond to the points of change, p_2,p_2, on the architrave. Finally, the circles at the base of the guttæ cast half ellipses inscribed in the half parallelograms $J_1f_1f_1$. The figure shows the construction.

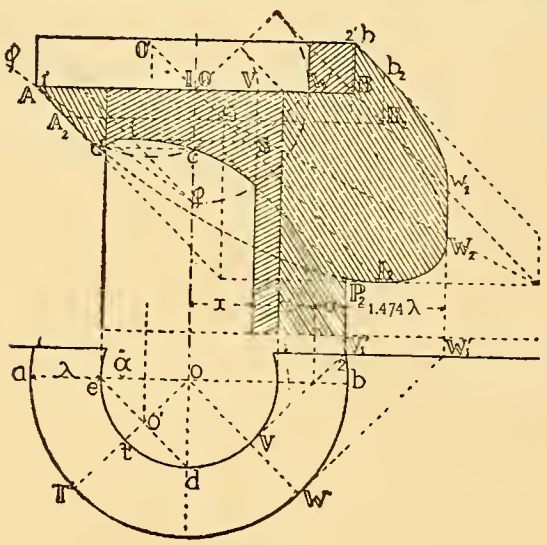

Fig. 50.

§ 21. Shadow of a Circular Cap on a Column. (Fig. 50.)

Axis of column $\frac{1}{3}$ R in front of wall.

(a) Shadow cast on the shaft.

First. Find in W and V the shades on the two cylinders.

Second. Point of loss. A circle of radius OW is the shadow of AB on the 45° oblique plane. The point where this arc crosses VS, since it lies in the surface of the cylinder and also in the line of the

auxiliary shadow, is a point of the real shadow and is also the point of loss. (Tangent at 45°, see § 3, Theorem of Point of Loss.)

THIRD. The highest point. The circle of auxiliary shadow cuts the axis at φ; it will be seen that this is the shadow of the 45° point of the fillet. Employing the method of cutting planes the 45° ray $t\varphi$ is revolved parallel to the wall on a vertical axis through φ and becomes the ray φ. The intersection of this line with the contour of the column enables us, by revolving back the ray to its first position, to locate t, the highest point of shadow.

The following will possibly be a simpler operation: Draw a 45° line from W to S on the line of shade V′V, and locate t on the same horizontal with S and at the same distance from the axis. (WS is a line in the oblique plane of auxiliary shadows.)

The highest point and the point of loss with their tangents generally suffice for drawing the curve.

FOURTH. Point c on the axis. The continuity of the curve already drawn gives this point; but it can be found directly.[1] In practice, however, it is unnecessary to find it directly.

FIFTH. c′ on the contour is, on account of symmetry, on a horizontal with c, but is also easily found as just suggested.

(b) Shadows on the wall. Knowing $\propto$, the distance of the axis from the wall, we find A_2B_2, shadow of the diameter AB, at a distance $\propto$ below AB. The ellipse of shadow $A_2I_2W_2$ is drawn as already shown in § 6.

The shadow of the column $P_2V'_1$ is at a distance from the shade line equal to $x + \propto$. x is the distance from the line of shade to the axis line.

REMARKS.—1st. P_2 is the shadow of the point of loss; 2d. W and W_2 are in the same 45° line; 3d. (important) The projection of the shadow of the fillet beyond the shadow of the column, or the distance $V'_1W'_1$, must have a breadth $\lambda' = 1.414\lambda$, or about $1\frac{1}{2}\lambda$, λ being the projection of the fillet beyond the shaft.[2]

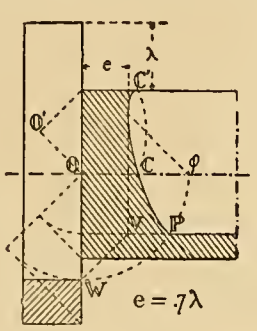

Fig. 51.

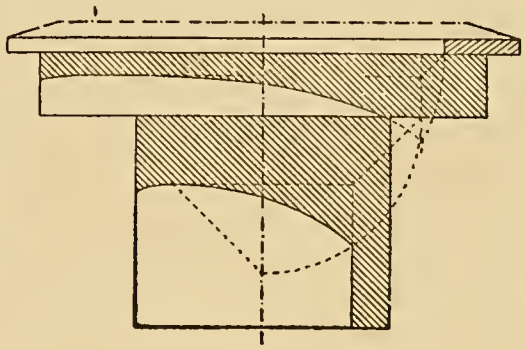
Fig. 52.

(c) Horizontal cylinder. (Fig. 51.) The important points, viz., nearest point, point of loss, point on the axis, and point on the contour, are found in the same manner as above.

The shadow shown in Figure 52 occurs sometimes in architecture, as in the ancient circular temple dedicated to Vesta. In the figure the exterior columns are omitted; the plan will be necessary in order to determine their shadows on the surface of the wall.

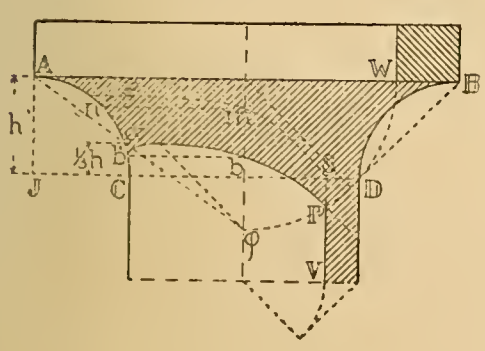

Fig. 53.

§ 22. SHADOW OF A CIRCULAR FILLET WITH A CONGÉ OF CIRCULAR CONTOUR. (Fig. 53.)

FIRST. Draw the shades W of the fillet and VS of the shaft. The latter is continued to a theoretical line of shade Smgn. n is at the 45° tangency on the contour; m is at the same level on the axis line; g is found from the φ tangency on the contour. S is on the shade line of the cylinder at the springing of the congé.

SECOND. Point of loss P, found as in the last paragraph.

THIRD. The point of greatest height $\propto'$. Draw through A the ray φ; take in $\propto$ its intersection with the congé and revolve it forward about the axis of the shaft through 45°, the point φ being fixed. (§ 21.)

---

[1] Both c and c′ may be found by means of the auxiliary shadow of the cap on an oblique plane under 45° passed through d and e (plan).

[2] In Fig. 50 it will be noticed that $V'_1W'_1$ ($=\lambda'$) is the hypothenuse of an isosceles right triangle of which $V'W'$ ($=\lambda$) is another side. Sine 45°=.7071 and $V'W'$=.7071$W'_1V'_1$; hence $\lambda' = \frac{\lambda}{.707}$ = about 1.5λ.

REMARK.—Fig. 54, drawn at a large scale, shows that the angle $\propto OA$ equals 2φ.[1] Let h′ be the height of $\propto$ above the springing, C, and let h (= OA = O$\propto$) be the height of the congé; then, from the triangle $\propto gO$, $g\propto = O\propto \text{Cos.}2\varphi$ or $h' = h\ \text{Cos.}2\varphi = .3333h$. Hence $h' = \frac{h}{3}$.

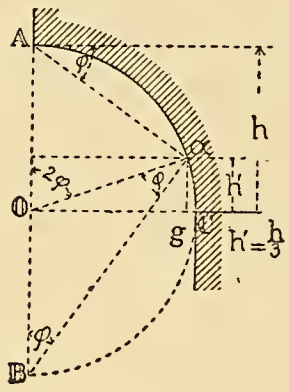

Fig. 54.

This result should be memorized, as it enables one to find the highest point of the shadow with very little drawing.

§ 23. SHADES AND SHADOWS OF AN ASTRAGAL. (Fig. 55.)

Suppose the axis of the column in the surface of the wall. If the axis were at a distance $\propto$ in front of the wall, the shadow on the wall would be displaced along a 45° line through a distance equal to the diagonal of $\propto$.

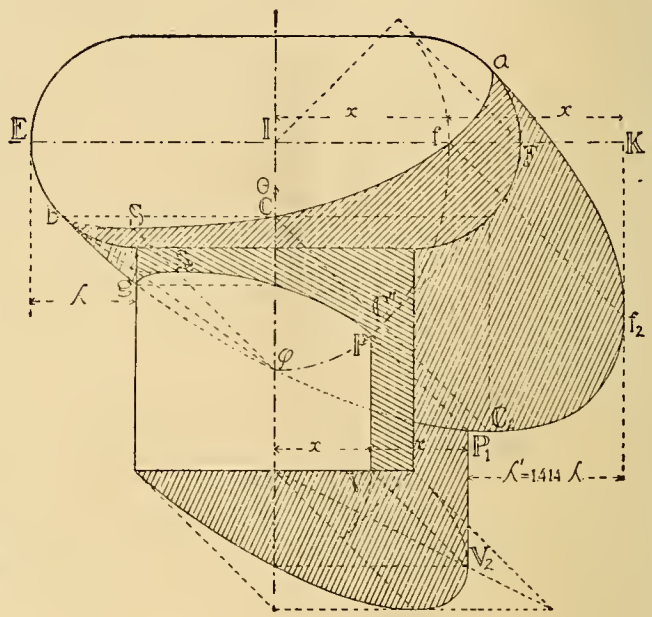

Fig. 55.

FIRST. Shade on the torus (see § 18). b and a are the points on the contour (45° tangency); a horizontal through b gives C on the axis line. The point f, on the equator, is found by circumscribing a cylinder about the torus; then If, or x, = .7 IF (see § 15).

SECOND. Shade on the shaft. It is found in PV as previously shown (§ 15). The shadow is P_1V_2, found at 2x from the axis.

THIRD. Shadow of the torus on the wall. The line Kf_2, containing the shadow of f at f_2, is at a distance from the axis equal to 2x (the distance of f from the wall is If; compare Fig. 50). Likewise C casts shadow in C_2, which is in a vertical with a (C is at a distance Cb from the wall). The point S, found by the line φ, casts shadow on the wall at φ. The shadow cast by the torus on the wall is then a curve passing through the following points: b (tangent at 45°); g (on the contour); φ (on the axis, tangent there is inclined at $\frac{1}{2}$); C_2 (tangent horizontal); f_2 (tangent vertical); a (tangent at 45°).

FOURTH. Shadow of the torus on the column. The point S_1, highest point on the shaft, is found by a φ line tangent to the contour of the torus. The point of change, P_1, projected back at 45° upon PV, gives the point of loss of the shadow of the torus on the cylinder. This shadow can then be drawn through P (tangent at 45°) and through S_1 (tangent horizontal).

The point on the contour (tangent vertical) will be in the same horizontal with the point where the shadow crosses the axis.

§ 24. TO FIND THE POINT OF LOSS DIRECTLY—THE OVAL CURVE, OR THE SHADOW OF A TORUS ON THE OBLIQUE PLANE AT 45°.

First find in f Pφ (Fig. 55) the shadow of the torus on an oblique plane passing under 45° through the axis. This auxiliary shadow we shall call the *Oval Curve* of the torus (it is not an ellipse). The use of this curve will be necessary in the following lessons.

Consider the oval curve as the envelope of the shadows cast on the 45° oblique plane by parallels of the torus; these parallels are circles and the method for finding their shadows has already been given (§ 12).

The equator EF (Fig. 55) casts a circle of radius If and determines one point f of the oval curve (tangent vertical). The parallel bC, at the 45° tangency, also casts a circle and gives a point C″ of the oval curve (tangent at 45°). Finally the point φ on the axis (determined by the line φ, tangent to the torus) is the shadow of the lowest point S; it is also a point of the oval curve (tangent horizontal).

---

[1] Consider the triangle A O$\propto$. The angle at A equals 90°—φ and, the triangle being isosceles, the angle at $\propto$ equals that at A. The sum of the angles of the triangle is 180°, hence the angle at O equals 2φ and O$\propto$ g equals 2φ.

This oval curve intersects the shade on the shaft at P, which is also in the oblique plane; then P is the point of loss.

REMARK 1.—In the practical application of this method the torus very often has a very small height in comparison with its breadth; in this case we may approximate the oval curve by a circle obtained as follows: Take a point θ at about $\frac{1}{3}$ of CI (Fig. 55), on the axis, and describe a circle with this point as centre. This circle may take the place of the oval curve without sensible error only on the condition that the height of the torus is very small in proportion to its breadth. (Fig. 56.)

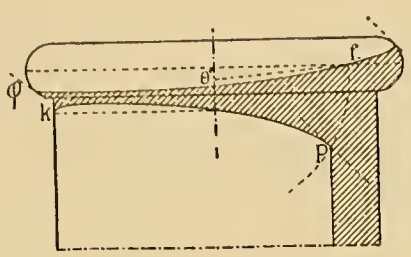

Fig. 56.

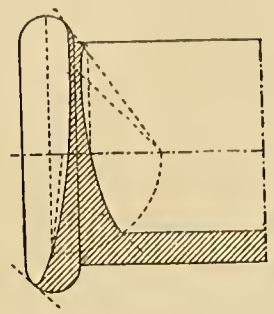

Fig. 57.

REMARK 2.—The astragal may, as in Fig. 57, accompany a horizontal cylinder. The shadows are the same, but are placed differently.

CHAPTER VI.

SHADOWS IN HOLLOW CYLINDERS, AND APPLICATIONS.

§ 25. Shadow of a Cylinder of a Steam Engine, in Section, with the Piston Projecting. (Fig. 58.)

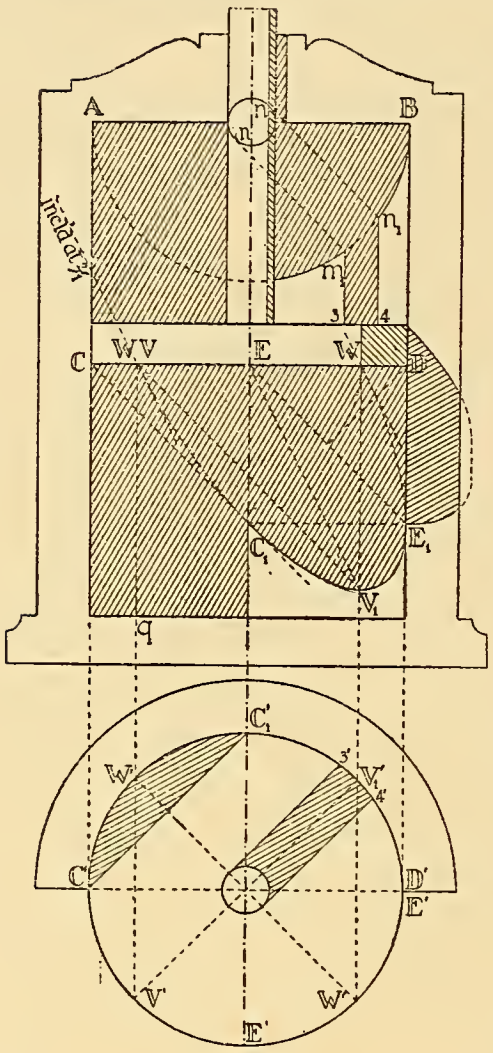

Fig. 58.

First. Shadow of the cap AB. The line AB, section of the cap, casts shadow in a semicircle described on AB as a diameter.

The shadow $m_1 3$ and $n_1 4$ of the shaft of the piston is found by drawing 45° tangents to a circle at mn. In reality the semicircle AB and the small circle mn serve as a plan.

Second. Shadow of the projecting piston. Find at W the shade of the piston and in Vq the theoretical shade of the hollow cylinder.

The shadow is cast by the projecting circle CD in the hollow cylinder, which has the same circle for its base. Now from the following theorem, "When two surfaces of the second degree (in intersection), cones, cylinders, spheres, etc., have a plane curve of entrance, the curve of exit is also plane" (Descr. Geom.), we may conclude that the shadow is a plane section of the cylinder and, consequently, is an ellipse. Two diameters of the circle at right angles, W′W′ and V′V′$_1$ for example, will give two conjugate diameters of the ellipse.[1]

VW and EV$_1$ are the conjugates of the ellipse corresponding to W′W′ and V′V′$_1$ of the circle, and EV$_1$ is inclined at $\frac{2}{1}$ (since V$_1$W=WV = 2EW); hence we may conclude that the tangents at the points of departure V and W of the ellipse are inclined at $\frac{2}{1}$, etc. Follow the figure.

Remark.—C casts shadow at C$_1$ on the axis (tangent at 45°) and E at E$_1$ on the contour[2] (tangent vertical).

§ 26. Shadow in an Open Cylinder, or Bridge Shadow. (Fig. 59.)

An application of the preceding. The half cylinder of Fig. 58 is here supposed open at the top. In this case we have only to deal with the portion VC$_1$ of the ellipse of Fig. 58.

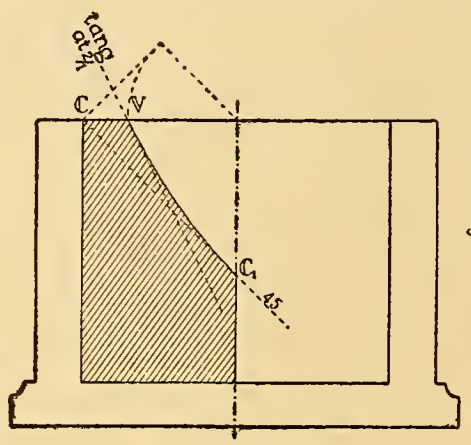

Fig. 59.

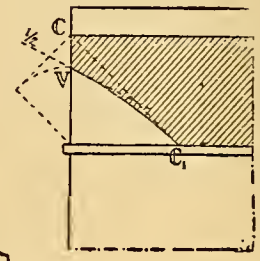

Fig. 60.

It will be noticed, 1st. That the point of departure V of the ellipse is the left 45° point of the semicircular end of the cylinder, and that at this point the tangent is inclined at $\frac{2}{1}$.

2d. That the point C$_1$ on the axis is at 45° from the point C, and that at this point the tangent is a 45° line through C. This enables us to draw the shadow.

If the cylinder has a horizontal axis, as in the case of a cylindrical bridge in section (Fig. 60), the line is the same; the point of departure is V. The tangent there is inclined at $\frac{1}{2}$.

---

[1] Two diameters of an ellipse or circle are conjugate when tangents at the extremities of one are parallel to the other.

[2] As C′D′ and E′C′$_1$ are also conjugate axes of the circle, EC$_1$ and EE$_1$ must be conjugate axes of the ellipse, hence the tangent at E is vertical and that at C$_1$ is at 45°.

The point C_1 is on the line of springing of the arch. The tangent there is at 45° from C. This shadow will be called the *Bridge Shadow.*

The shadow of an arch in section (Fig. 61) is merely a portion of the preceding shadow.

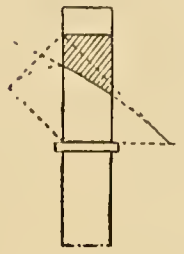

Fig. 61.

§ 27. SHADOW OF A RE-ENTRANT FILLET (SEMICIRCULAR PLAN)—THE ARCH. (Fig. 62.)

A half cylinder whose base is the semicircle AB; at the top a cover pierced by a smaller half cylinder CD; it is the latter which we call a re-entrant fillet. If the large cylinder becomes a vault, as in Fig. 63, the small cylinder will be called an arch.

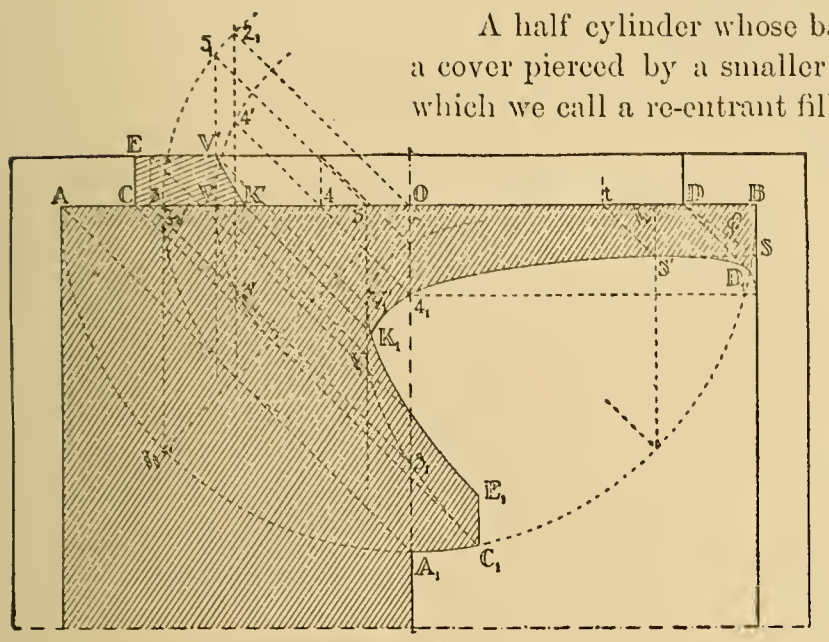

Fig. 62.

FIRST. The line AB will give as shadow the semicircle AA_1B, on which we find in C_1 the shadow of C, and in D_1 the shadow of D, where the line AB is broken.

SECOND. Shadow of the semicircle CD; this is the curve passing through the points C_1 (tangent at 45°); V_1 (tangent vertical, shadow of the 45° point of CD); S′ (highest point, tangent horizontal); and D_1 (shadow of the point D, tangent at 45°). (Note that these tangents are the shadows cast by the tangents to the circle CD on planes tangent to the cylinder.)

Further details of the operation are as follows:

(a) Point V_1, farthest to the left. It is the shadow of the 45° point, V, of the circle CD. But V is in the 45° oblique plane passing through the axis. Now, where is the vertical in the surface of the large cylinder, whose auxiliary shadow, cast back on the oblique plane, is 5_1V? It is the line $5V_1$; for the shadow of the semicircle AB on the oblique plane passes through $W5_1$, and 5 is the point on AB which casts 5_1; hence $5V_1$ casts shadow back to 5_1V and V_1 is the shadow of V on the surface of the large cylinder.

(b) Points 3_1 and 4_1 on the axis line. The same method may be used for any vertical of the large cylinder and any number of points found. Take the axis line; this casts shadow on the auxiliary plane in a vertical passing through 2_1. It crosses the shadow of CD at 3′ and 4′, which are the auxiliary shadows of 3 and 4.

(c) Highest point, S′. Found by the use of the ray φ.

§ 28. THE SAME PROBLEM—A SIMPLER METHOD (Horizontal Cylinder). (Fig. 63.)

Draw the two circles on the diameters AB and CD and use them as a section of the two cylinders revolved backward and to the right.

FIRST. V_1, the highest point. This is the shadow of V, the 45° point of the circle CD, and is found by projecting V upon the section in V′ and casting the shadow on the large cylinder in section at V'_1 and projecting this upon the elevation at V_1.

SECOND. Points 3′ and 4′ on the axis. These are found by cutting the two cylinders by a 45° plane passing through Z. This plane cuts the small cylinder in 3 (found by 3″ in section) which casts its shadow along the cutting plane on the line Z at 3′. Any number of such cutting planes might be used.

THIRD. t_1, point farthest to the left. This point is in a 45° cutting plane Ot'_1 (section) passing through the axis; the operation is evident from the figure. If the cylinders were not cut in half the complete shadow would be $E_1K_1t_1DE_1$.

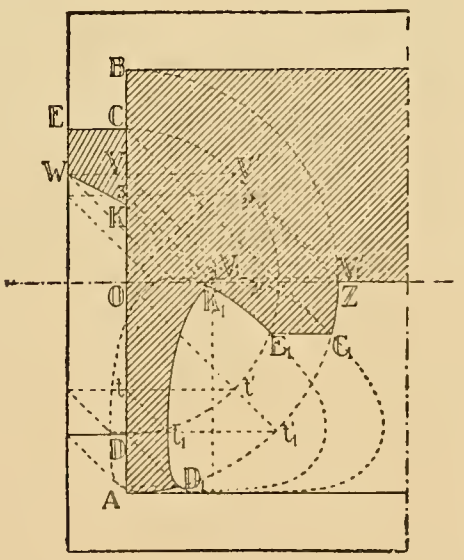

Fig. 63.

§ 29. ARCHITECTURAL APPLICATION.

In the corresponding architectural problem the lower half of Fig. 63 becomes vertical wall; ZC_1 and the line through t_1 become vertical lines; and C_1V_1 and E_1K_1 become arcs of ellipses.

CHAPTER VII.

SHADOWS IN HOLLOW SPHERES—SPHERICAL VAULTS.

§ 30. Shadow in a Bowl—In Plan. (Fig. 65.)

A hollow hemisphere forming a cavity in a cubical block.

The equator tku, in plan, casts a shadow tk_1u, which, by a theorem already referred to (§ 25), is a plane section of the sphere, and consequently a circle projected in the form of an ellipse.

First. Points of origin t and u. These are the points where the ray of light is tangent to the equator; tu is the major axis of the ellipse of shadow.

Second. The farthest point k_1. It is the shadow of k, located on the 45° meridian of symmetry; in order to find it use the method of cutting planes. The section made by a vertical plane through kg is revolved about kg into the plane of the equator and coincides with guk; the ray of light from the point k is shown in kk′, making the angle φ with kg; its intersection k′ with the surface of the bowl is transferred to k_1 by revolving the plane of section back to its first position.

Consider the triangle Ok_1k', in which Ok_1 is the minor axis of the ellipse of shadow and Ok′ is the radius of the sphere.

Now the angle at $O = 2\varphi$ as we may prove by considering the triangle Okk′, thus: Ok′ = Ok and angles k and k′ are equal, hence angle O equals $180° - 2\varphi$, so that the angle at O in the first triangle equals 2φ.

We have already seen that Cos. $2\varphi = \frac{1}{3}$. Then $Ok_1 = \frac{1}{3}$ Ok′ or $\frac{1}{3}$ R.

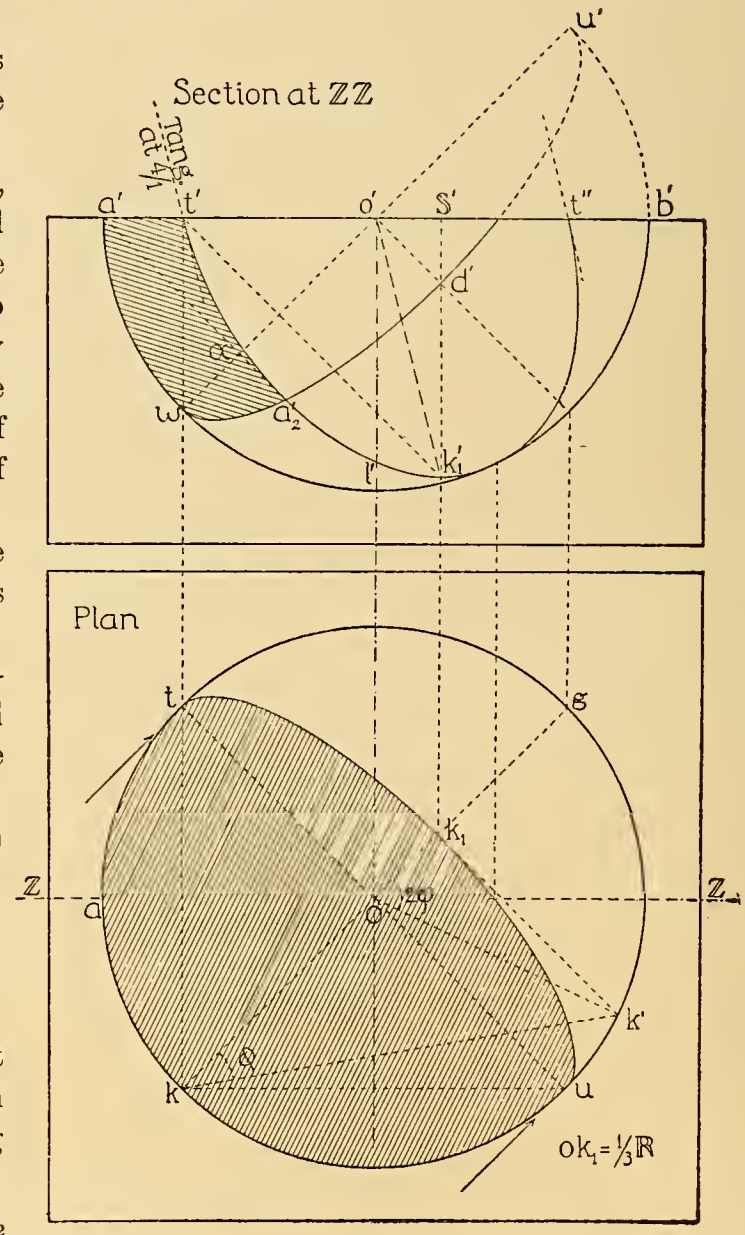

Fig. 65.

§ 31. The Bowl in Section.

In reality this is only a quarter of a hollow sphere.

First. If the meridian wa′u′ were complete it would give as shadow the ellipse wd′u′, identical with that found in the last paragraph, the minor axis o′d′ being equal to $\frac{1}{3}$ R.

Second. But the meridian stops at a′; we find in a'_2 on the ellipse the shadow of a′, and only the portion of the ellipse between this and w is used.

Remark.—We also have $\propto a'_2 = \frac{1}{3} \propto a'$, which gives the point a'_2 more exactly. When a circle is described on the major axis of an ellipse as diameter, the proportion of the ordinates of the circle to the ordinates of the ellipse is constant. Here the proportion is 1 to 3.

Third. Shadow of the equator. It is the shadow of the arc a t in the plan, and the ellipse of shadow is

the vertical projection of the ellipse tk_1u. Now a′ gives a'_2 (tangent at 45°); t′ is the point of origin (vertically over w); which is sufficient to determine the arc $t'a'_2$.

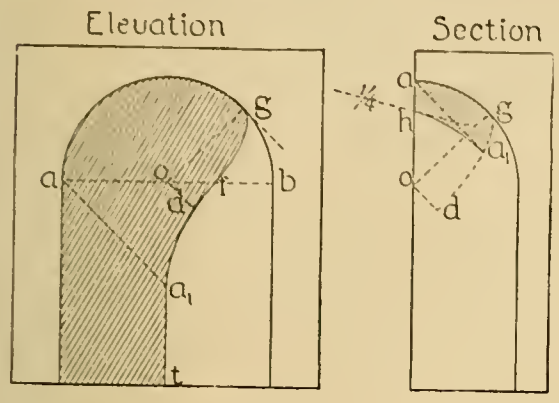

Fig. 66.

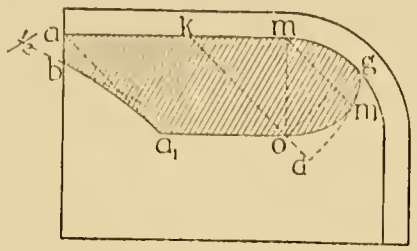

Fig. 67.

REMARKS.—1st. At t′ the tangent is parallel with $o'k'_1$, conjugate diameter of o′t′. (Compare plan.) But in the plan $ok_1 = \frac{1}{3}$ og; hence in the elevation $1'k'_1$ (= o′S′) = $\frac{1}{3}$ o′t″ = $\frac{1}{3}$ o′t′. Now t′S′ = S′k'_1 and o′S′ = $\frac{1}{4}$ t′S′ = $\frac{1}{4}$ S′k'_1. Hence at the point of origin t′, the tangent is inclined at $\frac{4}{1}$.

2d. In the bridge shadow the tangent to the ellipse at the point of departure is inclined at $\frac{1}{2}$. In the bowl in section the tangent at the point of departure is inclined at $\frac{4}{1}$. It is well not to confuse the two cases.

§ 32. SHADOW IN THE SPHERICAL NICHE. (Fig. 66.)

Elevation.—1st. Take od at 45°, equal to $\frac{1}{3}$ R. Draw the 45° tangent at g and the quarter ellipse ogd. It will only be used as far as f.

2d. Find shadow of a at a_1 (on the axis line).

3d. Connect f and a_1 by an irregular curve tangent to the ellipse at f and to the axis line at a_1.

Section.—The shadow is identical with that of the section in Fig. 65, but it is differently placed.

§ 33. CYLINDRICAL VAULT TERMINATING IN A HALF DOME. (Fig. 67.)

ba_1 is the bridge shadow. gm is an arc of the bowl ellipse. om is a part of an ellipse and is the shadow of the line km on the quarter dome, and tangent to a_1o at o and to the ellipse at m.

§ 34. SPHERICAL VAULT AT THE LEFT OF A CIRCULAR TUNNEL VAULT, WITH AN INTERMEDIATE ARCH—IN SECTION. (Fig. 68.)

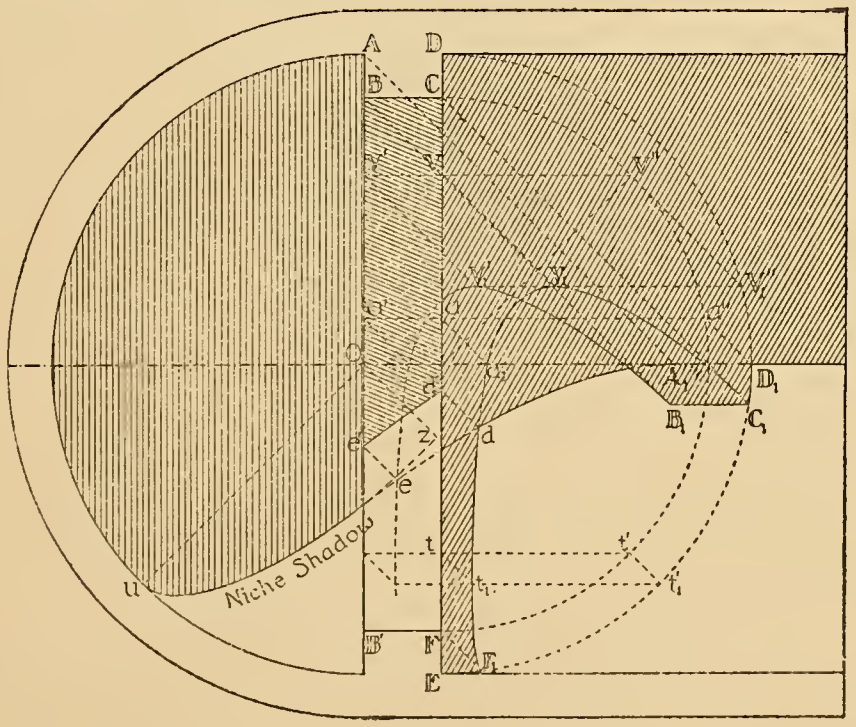

Fig. 68.

FIRST. Draw uedA_1, the niche shadow, as follows: Draw from u an arc of an ellipse whose minor axis oz equals $\frac{1}{3}$ of the major axis, and complete the curve to A_1 on the axis at 45° from A.

SECOND. Profile shadow at e′d′ on the arch. In order to locate d′ find, as in § 27, the shadow V_1dF_1 of the arch on the cylinder, and the point of change d; pass back to d′, and we find there one end of the interrupted portion of the shadow. The other end, e′, is derived from e, where the shadow of BB′ crosses the niche shadow. The remainder of the shadow, $B_1C_1D_1$, has already been given in § 27.

§ 35. THE SAME VAULTS REVERSED. (Fig. 69.)

FIRST. Find the shadow on the arch in the ordinary way (bridge shadow).

SECOND. In WZ we have the ellipse of the bowl shadow; the minor axis OZ is $\frac{1}{3}$ the major axis OW. It is of use only as far as A_1, shadow of A.

THIRD. The vertical line ACDB gives for its shadow in the sphere an ellipse AW′B, which is nothing else than a section of the sphere by a vertical plane at 45° passing through the centre. Hence we conclude that W′ is vertically below W.

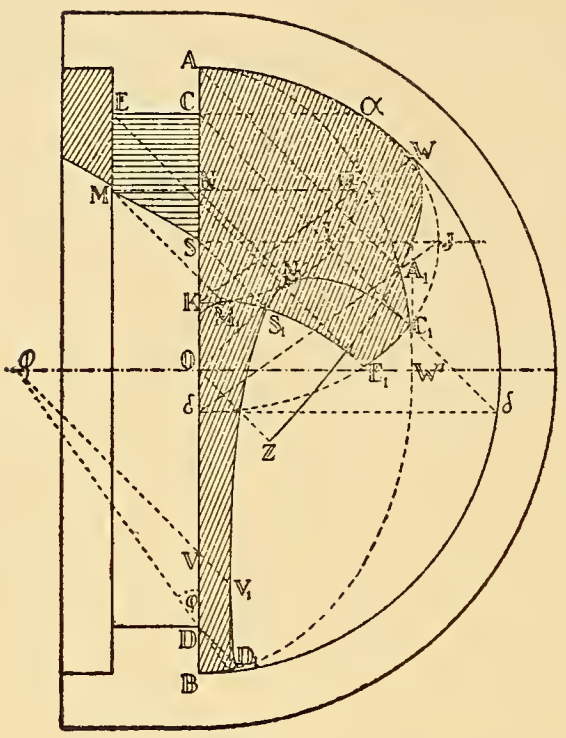

Fig. 69.

A_1, already found, must also be on this ellipse. The ellipse is used only between A_1 and C_1, shadow of the point C, and between B and D_1, shadow of the point D.

FOURTH. The horizontal line EC, of indefinite length, will cast as shadow an ellipse, $\propto JC_1\delta'$, which is the projection of the circle in which the plane of shade from this line cuts the sphere. This plane of shade is parallel with the ground line and is at 45° with both planes of projection.

It is easily demonstrated (a) that this ellipse is tangent to the profile of the sphere at $\propto$, where the line EC continued meets the profile; (b) that, if we draw the line $C\delta$ at 45°, the point δ, transferred to δ' on CD, gives in $S\delta'$ the semi-minor axis of the ellipse; (c) that, if we drop a perpendicular OI upon $C\delta$, the major axis is SIJ, and further, SJ equals $I\delta$[1]; (d) This ellipse must pass through the point C_1, already found.

Of this ellipse only the portion C_1E_1 is used.

FIFTH. Shadow of the arch on the sphere. The point V_1, farthest to the left, is found by means of the ray φ.

The shadow of the profile circle DC passes through the following points: D_1, already found (tangent at 45°); V_1, already found (tangent vertical); and C_1, already found (tangent at 45°). It is well to have an intermediate point, as at N_1, shadow of the 45° point of the circle, thus:

Find the shadow cast on the sphere by the horizontal MN. This will be, as in the case of EC, an ellipse which is the projection of a circle parallel with the ground line and inclined at 45°. The point $\propto'$ on the line MN produced, and in a vertical with $\propto$, already found, is a point of this ellipse; for it is the point where the line MN pierces the sphere.[2] $N\propto'$ will be the semi-major axis of the desired ellipse, and, as it must be similar to the ellipse $J\delta$, already found, we may draw $\propto'K$ parallel with $J\delta'$ and we shall have in K the extremity of the minor axis.

Or $\propto'K$ (also $J\delta'$) is inclined at the angle φ (easily demonstrated). This ellipse once drawn, we may find in M_1 and N_1 the shadows of M and N; and the shadows cast by the two circles EM and CN are tangent to this auxiliary ellipse.

Note that the point of change, S_1, is at 45° from the limiting point S.

---

[1] Consider the circle AWB as the profile section of the sphere, revolved parallel to the vertical plane, and $C\delta$ the plane of shade from EC$\propto$ in profile. Then the major axis of the ellipse will be found where a great circle of the sphere will cross the plane of shade at right angles. The only great circle that crosses the plane of shade ($C\delta$) at right angles is OIW. The line in which the plane of shade cuts the sphere is in reality a circle, and $I\delta$ is half the breadth of this circle. Hence SJ must be made equal to $I\delta$.

[2] To make this clearer we suggest that if the circle CD be displaced horizontally and parallel to itself until C takes the position $\propto'$, the circle CD thus displaced will be entirely in the surface of the sphere.

§ 36. Shadow of a Spherical Dome Supported by Pendentives and Arches. (Fig. 70.)

First. Construction of the vault. Imagine a hemisphere whose equator passes through the four corners of a horizontal square. Cut this sphere by four vertical planes passing through these four corners; they intersect the sphere in four circles. A short distance below these circles suppose the vertical planes cut by four horizontal cylinders; the cylinders are barrel vaults, which the sphere serves to join.

The spherical triangles which come to a point at the corners on the equator of the sphere are called pendentives, and through these the weight of the vault is transmitted to the supports. Over the pendentives is the cupola Wb.

Second. Shadows of the section. Suppose that the arch facing us is filled to half its depth by a wall. The three other arches are open.

(a) By our knowledge of the distances we find in the ordinary way the point O_1, shadow of the centre O on the plane of the face of the arch, and O_2, shadow of the same centre on the plane of the wall. From these points as centres, with the radius Ob, describe the two arcs 1–2 and 3–4.

(b) We have in $K_1a_1b_1$1 the known shadow of the arch that is in section (§ 7, Fig. 23).

(c) In vK we have the bridge shadow (§ 25 and § 26).

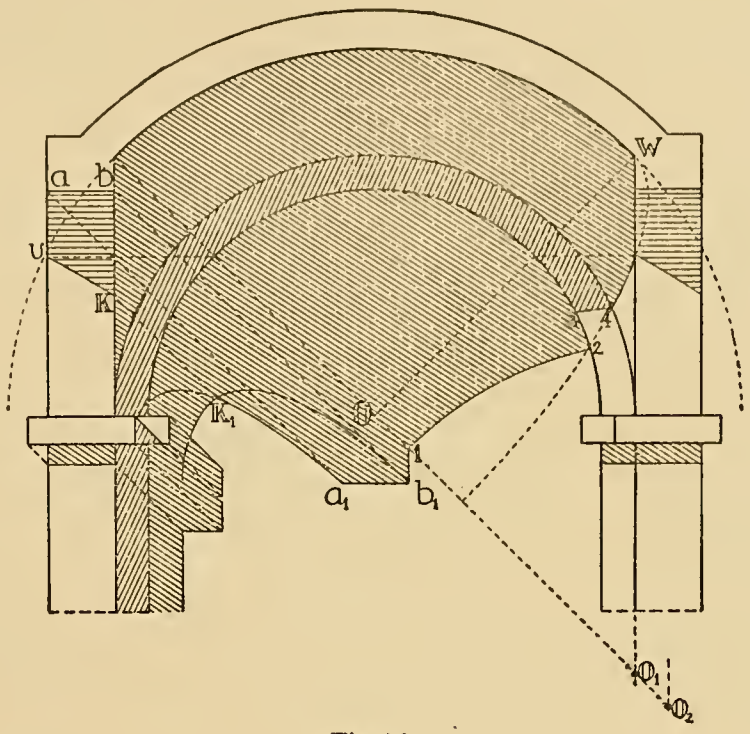

Fig. 70.

(d) From W construct the bowl shadow; verify by the point 4, which is common to the ellipse and to the arc 3–4.

CHAPTER VIII.

VARIOUS APPLICATIONS.

§ 37. Doric Capital. (Fig. 71.)

(a) Shadow of the cyma on the abacus. No difficulty. (See § 8 and § 9.)

(b) Echinus. This is a portion of a torus whose section is not an arc of a circle. The profile is a free-hand curve.

First. Find the shade on the echinus as follows: F, 45° point of the equator; B, on the left contour (45° tangency) and B′ on the axis line at the same level with B; C′, lowest point, found by use of the line at angle φ.

Second. Oval curve of the torus (see § 24). The useful part of this auxiliary shadow passes through the points F (tangent vertical), B_2 (tangent at 45°), φ (tangent horizontal); we shall call this curve the oval curve of the echinus.

(c) Shadow of the abacus on the echinus. The edge of the abacus AA casts shadow on the auxiliary oblique plane in a 45° line, AI. The shadow of AA on the echinus is no more than a section of the echinus by a plane passing through AA, the plane being inclined at 45° and parallel with the ground line.

If this plane were to turn upon the axis of the capital through 90° it would become perpendicular to the vertical plane of projection and would there be represented in profile by the line AI. This would be the plane of shade cast by the edge shown in profile at A.

Now the shadow MgB is the profile projection of the shadow M′pB′ and M′ may be located on the axis in a horizontal from M.

In order to find the other points use the method of oblique projections as follows: The two points of loss P′ and p are found from the intersections of the auxiliary shadows of the echinus and of the abacus in P'_3 and P_3. Tangents at p and P′ are at 45°.

Remark.—By reason of the symmetry of the section through M′ the point p″, opposite P′, will be at the same distance from the axis as P′.

As g is the profile projection of p the two points must be in the same horizontal.

(d) Shadow of the abacus on the necking and on the fillet. From a point I as centre, first with a radius equal to that of the necking, draw an arc of a circle, A_2g_2; this is the shadow of the line AA; then take a radius equal to that of the fillet, and with the same centre, I, describe another arc, g_1p_2, the shadow of AA on the fillet. This arc is limited by p_2, shadow of the point p, and by g_1, the point that casts shadow in g_2.

1st. Shadow on the fillet. p_2, already found, is one point. The limiting point h′ is found by drawing kh_3, the auxiliary shadow of the fillet, on the oblique plane. (This is a circle passing through k, the 45° point of the edge of the fillet.) Take its intersection h_3 with the oval curve of the echinus and locate h′ by a 45° line.

2d. On the necking. Find first, as shown above, $g_2h_2\pi$, shadow of the fillet on the necking (π is the point of loss). This curve is of use only between h_2 and g_2 and near the left contour.

At the point h′ the shadow of the echinus drops from the edge of the fillet and follows the curve $h_2\sigma$. σ is the point of loss, and is at the point where the oval curve crosses the shade line of the necking. The tangent there is at 45°, which enables us to draw the curve $h_2\sigma$.

(e) Shadow on the astragal. Find in $\propto \theta_1$ the oval curve of the astragal. (We may here, without sensible error, substitute for the true oval curve the arc of a circle with centre at ω; see § 24.) Take its

intersection with the oval curve of the echinus at θ_1, and from it derive θ', the point of loss of the astragal; this enables us to draw the shadow of the echinus on the astragal in $\sigma_1\theta'$; at θ' the tangent is at 45°.

The remainder of the construction is easily understood.

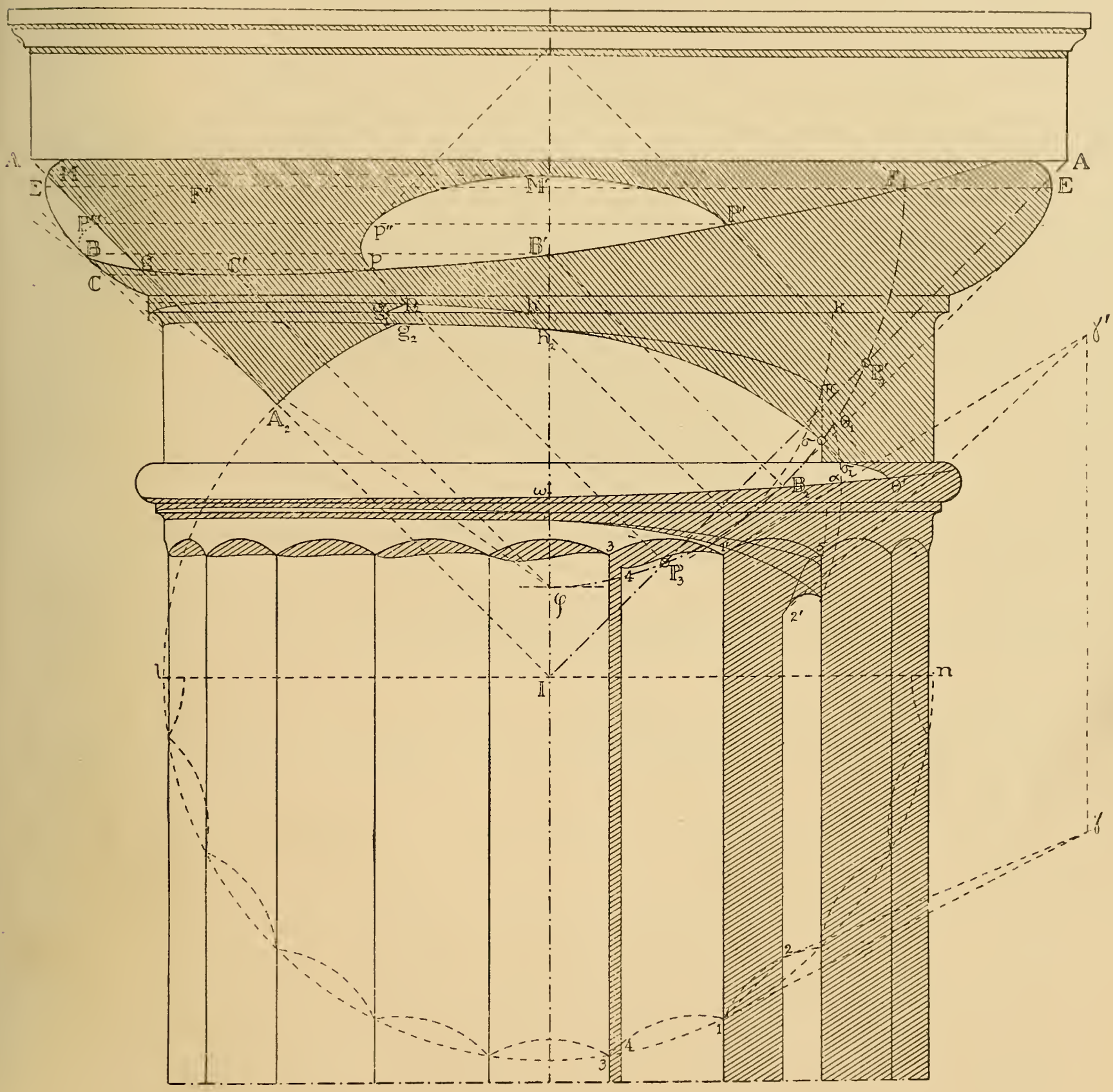

Fig. 71.

(f) Shadows in the flutes. A plan of the shaft is necessary, as at n3l. Notice that the line $2'\gamma'$ is nothing else than the shadow of $1'\gamma'$, (tangent to the head of the flute at 1') on the plane 2γ, which is tangent to the hollow surface of the flute. It follows that, as $1'\gamma'$ is tangent to the line which casts the shadow, $2'\gamma'$ will be tangent to the line of shadow.

§ 38. SHADOWS OF A DORIC COLUMN ENGAGED IN A WALL. (Fig. 72.)

Suppose the column engaged $\frac{1}{3}$ of its diameter; then the axis is distant $\frac{1}{3}$ R from the wall.

FIRST. Shadows on the capital. See the preceding paragraph.

SECOND. Shadows on the wall. (a) The shade on the shaft of the column is distant x (= .7 R) from the axis line and its shadow is distant x + d from the line of shade.

(b) A is a distance AO + d from the wall, and A_1 is easily found in a vertical at a distance AO + d to the right of A.

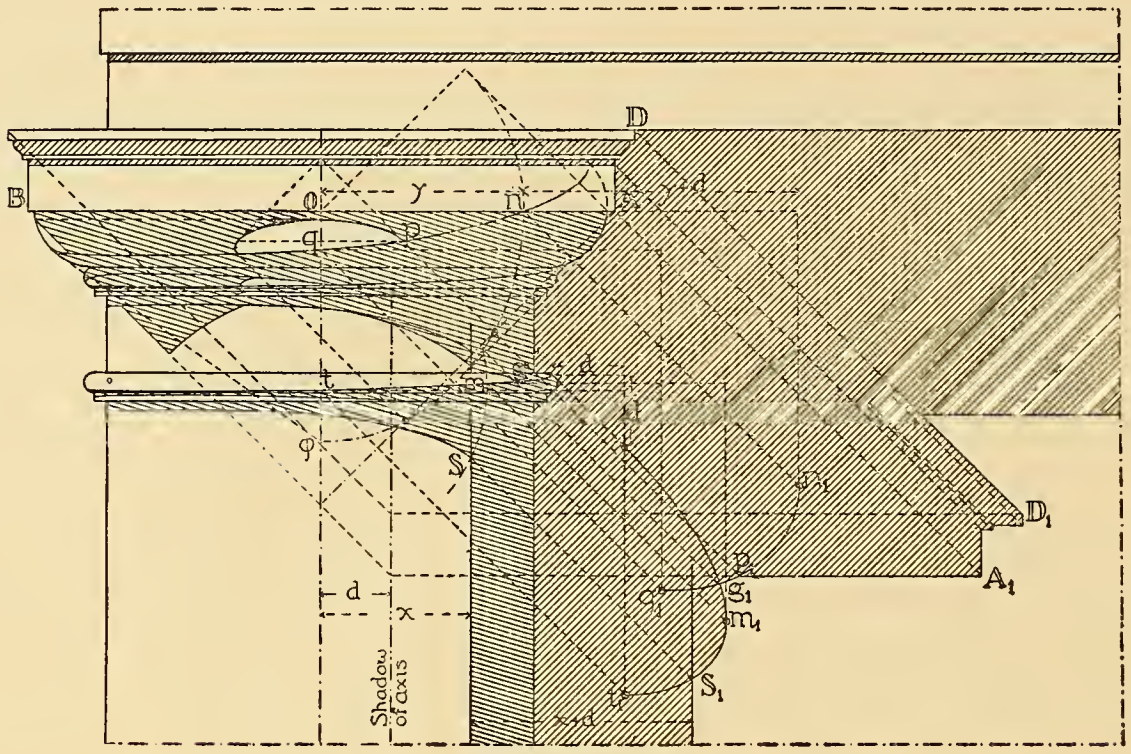

Fig. 72.

(c) The astragal gives a shadow $g_1 m_1 S_1 t_1$, found as follows: The point farthest to the right, m_1, is the shadow of m, the point where the shade crosses the equator; t_1 is shadow of t; g_1 and S_1 are derived from the points of loss g and S.

(d) The echinus casts a shadow whose point of change is p_1, shadow of the point of loss p. The point farthest to the right is n_1, shadow of n; n is on the diagonal of the abacus, and hence n_1 is on a diagonal of the shadow, drawn from A_1 at an inclination of $\frac{1}{2}$. The actual shadow is only the portion between p_1 and g_1, the latter being derived from the point of loss on the astragal. m_1 and n_1 may be found perhaps more easily by considering their actual distance from the wall.

§ 39. SHADOWS OF A PEDIMENT. (Fig. 73.)

(a) Description. The moldings of a pediment are drawn by taking the horizontal moldings of the cornice and inclining them to the slope of the pediment. The crown molding is used only on the slopes of the pediment.

The projections from the wall of the several parts of the cornice are taken from the right section on a strip ZZ; place this strip horizontally at any convenient place above the pediment (as at Z′Z′) and by means of it find the line in which a vertical plane which is parallel with the rays of light will intersect the pediment. It will be readily seen by referring to a plan that in this section the points A,B, etc., will appear, in the elevation, to be the same distance from the line Z′b as they are from the wall; hence in drawing this section the actual projections of the parts from the wall may be used.

A ray of light passing through A is in the plane of section, and the shadow of A is a, at 45° on the line in which the plane cuts the moldings. In the same way B casts shadow at b.

The plane of shade cast by the line BD cuts the wall in bb′; the line in which this plane of shade cuts the moldings is found as follows: Imagine a plane which is parallel with the wall and in contact with the outer edge of the ovolo. Then the point e′ should be in the line in which this latter plane is cut by the plane of shade; e″, on Bb and in a vertical with e, is one point of this line, and it enables us to locate e′. The other points between e′ and b′ are found in the same way. Now a ray of light through D lies in the plane of shade cast by BD, and the shadow of D is d′, at 45° on the line in which the plane of shade cuts the moldings; this determines the position of the shadow cast by DD. The dentil course and the cyma-reversa cast similar shadows.

The shadow of the crown molding is cast by exactly the same process. The line in which the plane of shade from AC cuts the crown molding of the right side, Cm′, may be accurately constructed by points just as the curve at A may be constructed from the right section. Choose any point m_1 on the right section; project to the strip ZZ at m_2; m is thus easily located in the oblique section. m′ is found in exactly the

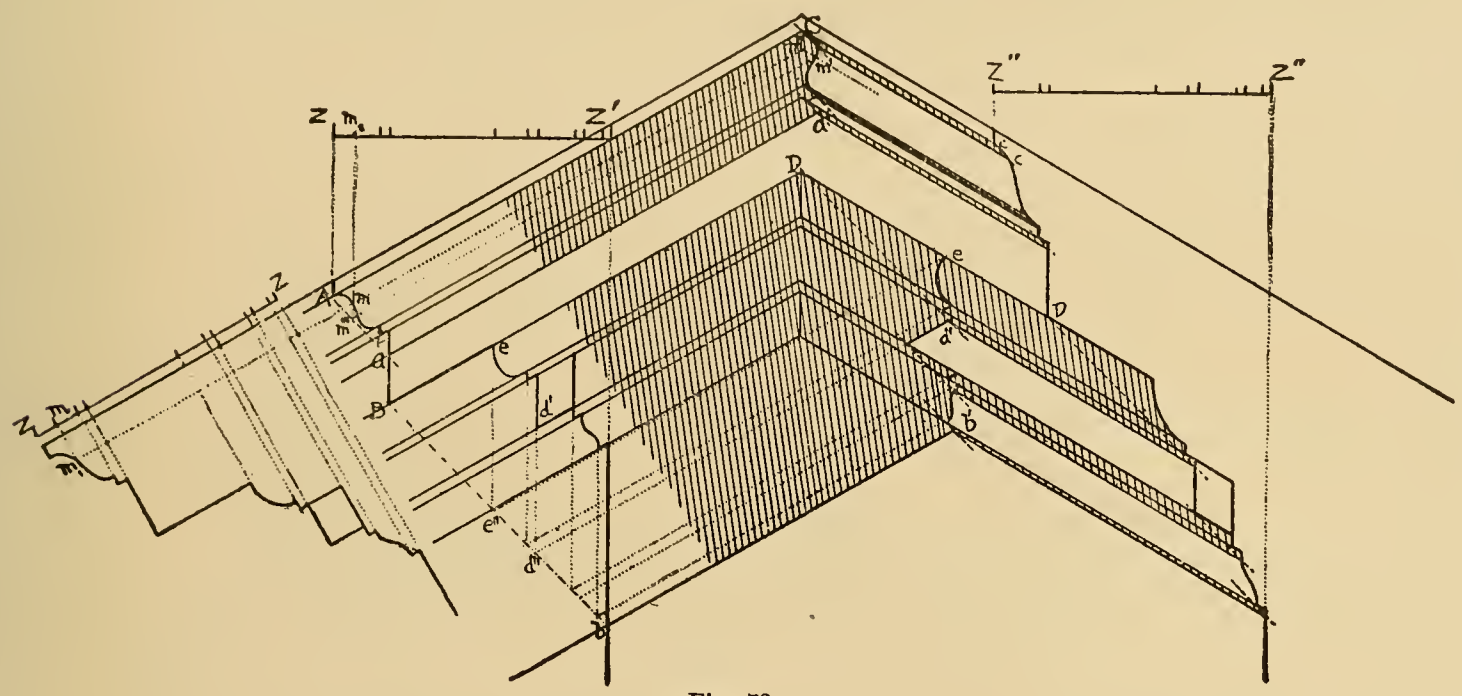

Fig. 73.

same way in which e′ was found, mm″m′ being the line in which the molding is cut by the plane parallel with wall.

The remainder of the construction is evident.

It may sometimes be convenient to find the shadows on the right side of the pediment independently of those on the left side. The same principle applies as indicated at Z″Z″.[1]

§ 40. SHADOWS ON THE ATTIC BASE. (Fig. 74.)

I. To find the shades. 1st. ab is the shade on the shaft; $b\beta$ (dotted line) is the shade on the congé, β_i is the point where a 45° line is tangent to the contour of the congé. This is a theoretical shade line in the greater part of its length; c is the shade on fillet No. 1.

2d. K′kg′ is the shade of the small torus (§ 18).

3d. e is the shade on fillet No. 2, and q is the shade on fillet No. 3.

4th. rS_1 is the shade of the large torus.

II. The shadows.

(A) Auxiliary shadows cast on the 45° oblique plane by the lines which cast or receive shadows.

1st. As the fillets and the shaft are cylinders their lines of shade lie in the oblique plane.

2d. The circles of the fillets give for their auxiliary shadows other circles passing through their 45° points, thus: (a) the circle of fillet No. 1 gives the arc cf_3; (b) the circle No. 2 gives the arc $e\infty_3$; (c) the circle No. 3 gives arc qP_3.

3d. The torus gives the oval curve $\varphi_1 h\varphi$, found as indicated in § 23; the large torus gives the oval curve rSW, of which we draw only the upper part.

(B) Actual shadows. The intersections of the auxiliary shadows enable us to determine points of the real shadows. Thus:

The auxiliary circle cf_3 crosses at f_3 the shade of the shaft. The ray of light f_3f' gives in f′ the limiting

[1] This treatment of the pediment shadow is chiefly derived from *Elementary Shades and Shadows*, by Mr. Lawrence, of the Institute of Technology, and is substituted for the method given by M. Pillet on account of its greater simplicity.

point of the shadow of the shaft on the congé; the shadow then falls from f′ to f″ on the small torus. t′ is the point of loss of the shadow of the fillet on the torus.

The oval curve $h\varphi$ of the small torus crosses the shade of the shaft at g_3. The ray of light g_3g' gives in g′ the point of loss of the shadow of the shaft on the small torus (tangent 45°).

Likewise the auxiliary shadow qP_3 of the fillet No. 3 crosses the oval curve $\varphi_1h\varphi$ of the small torus at P_3; and the ray of light P_3P_2 gives in P_2 the limiting point of the shadow of the small torus on the circle

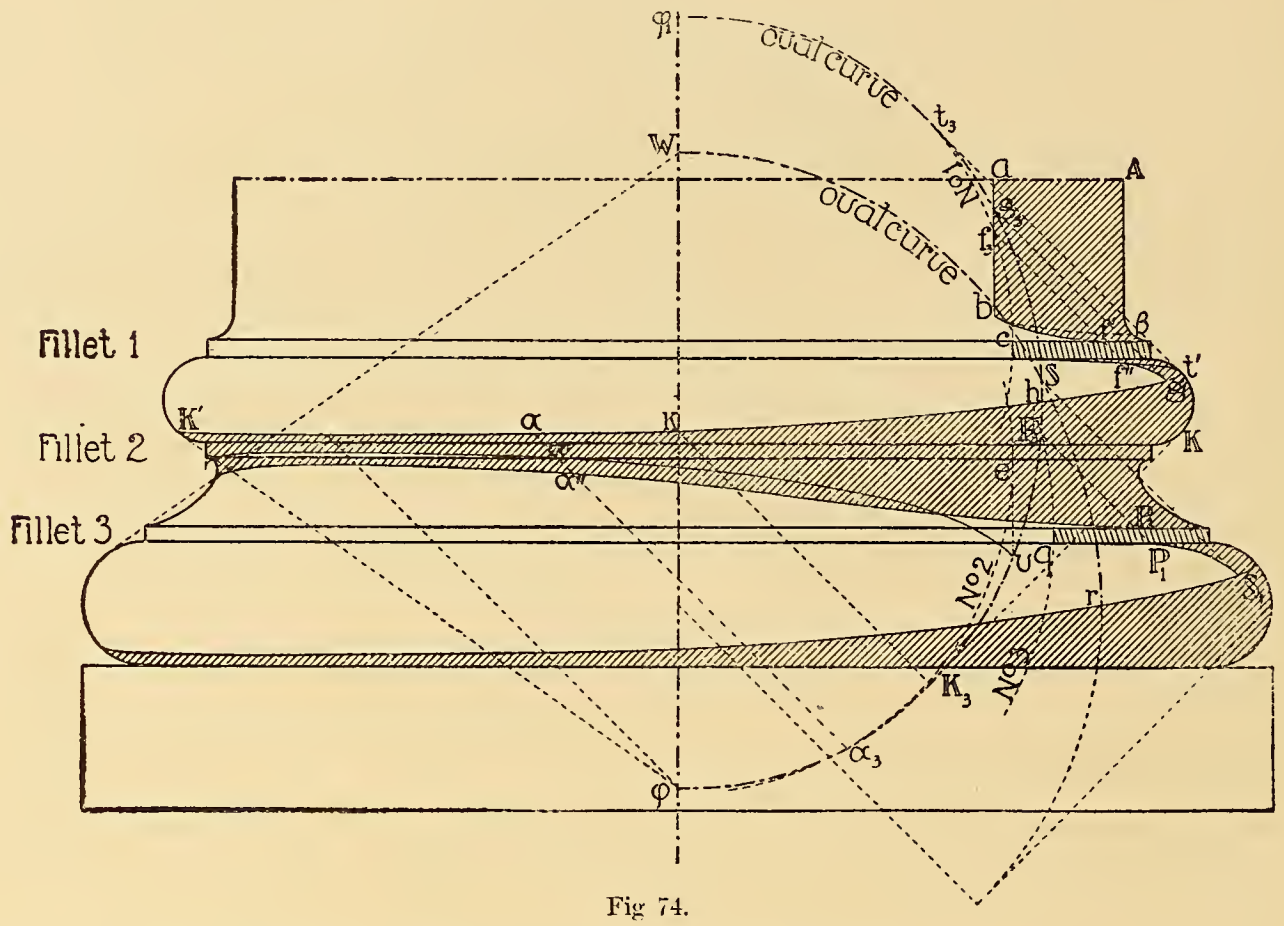

Fig 74.

of fillet No. 3. The shadow then falls from P_2 to P_1 on the large torus.

The oval curves $\varphi_1h\varphi$ and WSr intersect in S; hence the ray of light SS_1 gives in S_1 the point of loss of the shadow of the small torus on the shade of the large torus. v is the point of loss of the shadow of the small torus on the cylinder of which fillet No. 2 is a part, etc.

§ 41. THE CORINTHIAN CAPITAL. (Fig. 75.)

There is no truly simple way of finding the shadows of the Corinthian capital, because of the leaves that ornament the bell; but it is possible, by considering only the "bells" which enclose the leaves, to determine the chief masses of the shadows; the details may then be filled from the imagination.

FIRST. Shadow of the abacus on itself. We can if necessary treat the shadow of the abacus as the shadow of a "re-entrant fillet," but it is preferable to make use of the plan. Points of the shadow may be found as indicated at 55′, casting shadow at $5_15_1'$.

SECOND. Shadow of the abacus on the bell. The point AA′, which is in the 45° meridian of symmetry, gives a shadow A'_3, found by the use of the angle φ revolved into the vertical plane and back again (as at A_1A_1'.)

In order to find the limiting points α'_1 and β'_1 on the circle 4, which forms the edge of the bell, use the method of oblique projections, applied by taking as the auxiliary plane of shadows the horizontal plane of the curve $\alpha'\beta'$. Thus the circle 4, which forms the edge of the bell, is cast back along 45° rays upon the lower plane of the abacus; the operation is shown in the plan. Cast from O backward at 45° to O_1; with O_1 as centre describe a circle equal to the circle 4; this cuts the lower edge of the abacus in α and β;

transfer α and β to α' and β' on the elevation and thence deduce, by 45° lines, the desired points α'_1 and β'_1.

THIRD. Shadows of the bells enclosing the leaves. These shadows are entirely analogous to the shadow of the astragal (§ 23).

These masses of shadow once found, **we can, by drawing** the leaves in their enclosing bells, fix upon

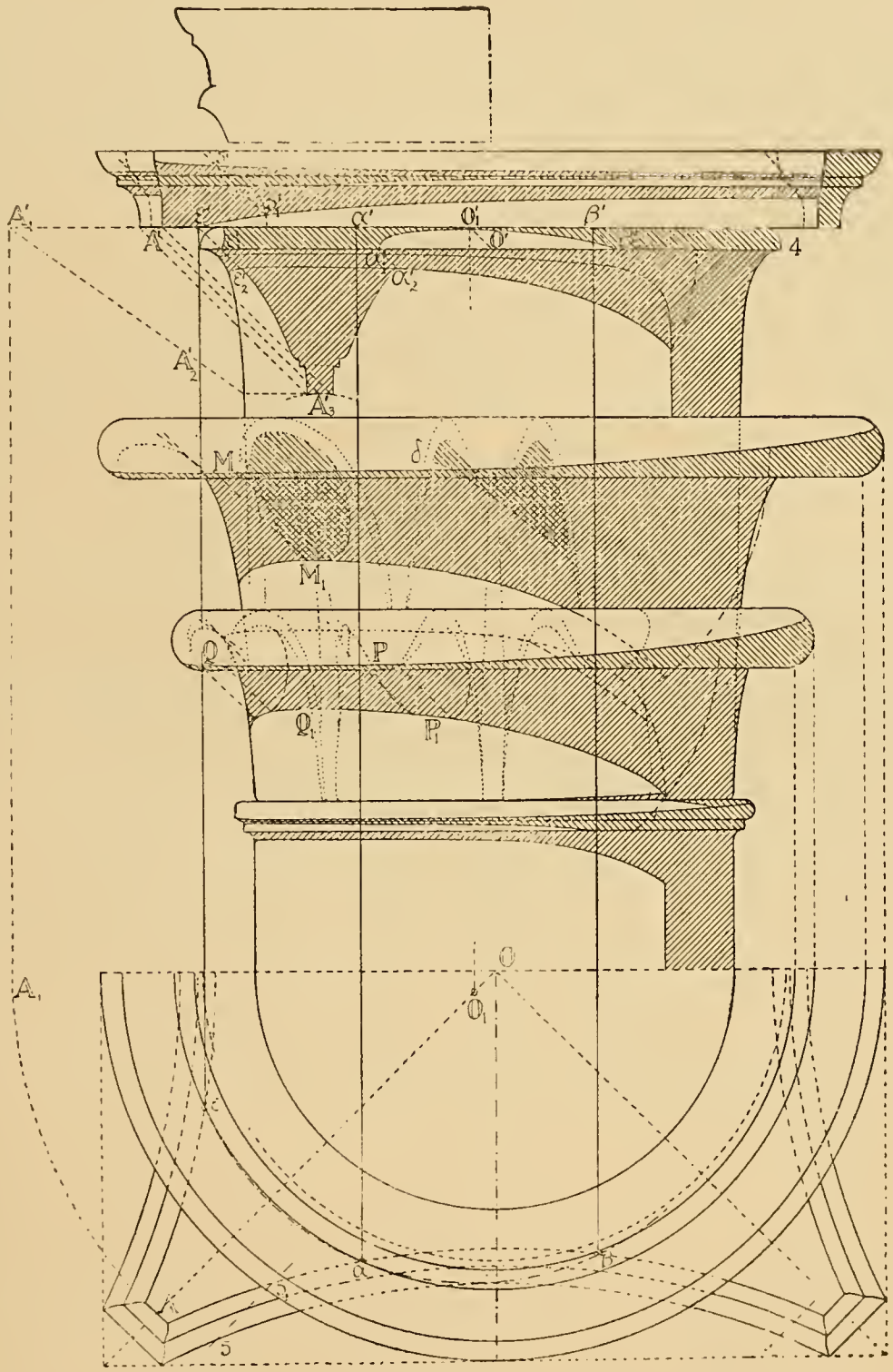

Fig. 15.

such points as Q_1,M_1, or P_1 of the shadows of the leaves; the remainder is completed from the imagination; but in order to be successful in this it is quite necessary to have drawn and shaded several capitals of this kind from nature lighted from a direction approaching as near as possible to that of the empirical ray.

§ 42. SHADOWS OF A GREEK IONIC CAPITAL—DRAWING THE CAPITAL. (Fig. 76.)

A Greek Ionic capital, similar to that of the Erechtheium at Athens, is in reality composed of two parts, viz.:

FIRST. A necking and an echinus, as in the Doric capital.

SECOND. A cushion, replacing the abacus of the Doric order; this cushion, consisting of the abacus and

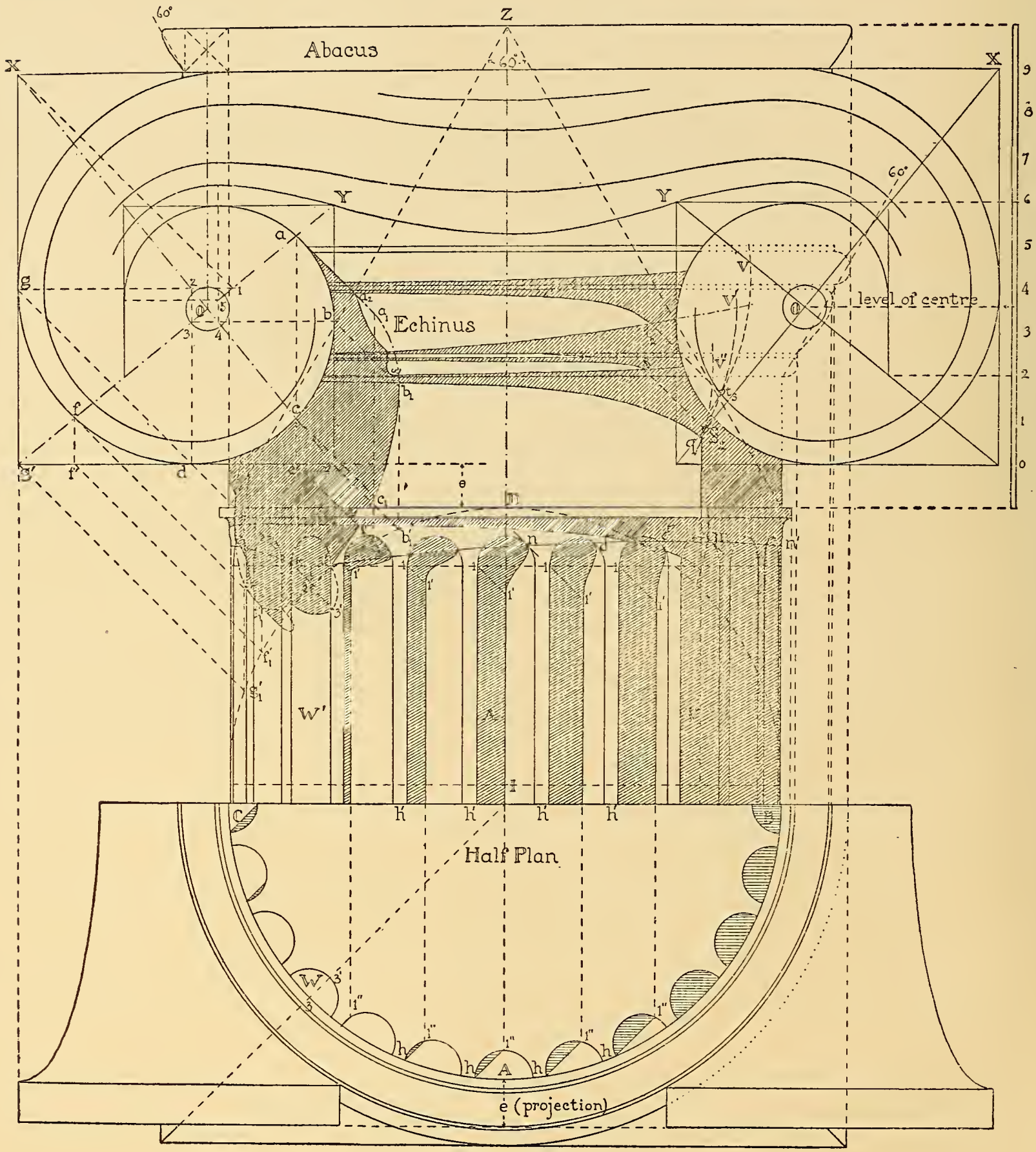

Fig. 76.

the volutes, fits very closely upon the echinus owing to the two spreading projections which constitute the volutes.

Separating the two parts is a small half-round in the form of a torus, analogous to the straw cushions used by stone carriers to prevent the sharp edges of the stones from crushing. In reality, this half-round is of the same body with the cushion and is cut out of the same stone.

Without entering into details, we give the following suggestions for drawing this capital:

1st. The height ZD equals the altitude of an equilateral triangle constructed on the diameter of the shaft.

2d. The moldings above and below the echinus are constructed on lines at 60°. (See dotted profile at the right.)

3d. The total height of the capital is divided into eleven equal parts, and the height of the volute is nine of these parts.

4th. The centre O of the eye divides the total height of the volute in the proportion of 2 to 3.

5th. By the use of horizontal and vertical tangents the volutes may be drawn freehand without the use of the compasses.

6th. The volutes project in front of the face of the column a distance e (see plan) equal to $\frac{1}{9}$ the height of the volutes, or equal to the height of the abacus.

§ 43. Construction of the Shadows.

(A) Shadow of the volute on the shaft. For the present consider the cylindrical surface of the shaft alone, ignoring the flutes and the echinus. The shadow on the bare shaft is the curve $f_1c_1b_1a_1$, found as follows: 1st. The horizontal tangent g′b′ casts shadow in the circle $g'_1b'_1$, whose centre is I (found as described in § 19), since the distance e (see plan) of the face of the volute in front of the face of the column gives the point D of the circumference.

2d. The points g′f′b′ and c′, projected at 45° upon this circle, permit of locating the shadows of the most important points of the volute, viz.: shadow of f in f_1 (tangent at 45°, f being the 45° point of tangency on the contour, § 3); of d in d_1 (the circle and the curve of shadow have there the same tangent); of c in c_1; of b in b_1 (tangent vertical); and of a in a_1 (tangent at 45°, § 3). We shall retain this curve, though it is only the shadow on the surface of the column, and break it over the other surfaces of the capital after having found their shades and shadows.

(B) Shadows of the surfaces of revolution. There is nothing new to be said regarding these.

1st. Determine the shades on all the tori and on the shaft.

2d. Draw the oval curves of the different tori, viz.: Vt_3, oval curve of the half-round; $V't_3q$, oval curve of the echinus; $V''S_2$, oval curve of the round below the echinus; in this last case the arc of a circle may be substituted for the oval curve, as already described.

3d. By means of these, passing back along rays of light, we find at t_2 and at S the points of loss (tangents at 45°).

4th. Then break the shadow of the volute over the moldings, always taking care to place a limiting point (such as 1′ on the astragal), or a point of loss (such as ω), on the same 45° line with the corresponding point of change.

This interruption to the line of shadow is drawn by the imagination.

(C) Shadows on the flutes. 1st. The middle flute A gives the niche shadow treated above. The point of departure n is at the 45° tangency, and the point of junction, 1′, is found on the shadow cast by the left edge of the flute.

The shadows in the other flutes are found as follows:

2d. By the plan we may find in 1″, 1″, etc., the shadows cast by the left edges on the backs of the flutes; these give in the elevation all the points of junction, 1′, 1′, 1′, etc.

3d. The points of departure δ,δ,m, are not found as easily as the point n of the central flute. But we can easily find four of these points, viz.:

(a) In the flute WW′, at 45° facing the light, the point of origin of the shadow is k, which is the springing of the ellipse; for if the sphere, of which the head of the flute is a part, be circumscribed by a

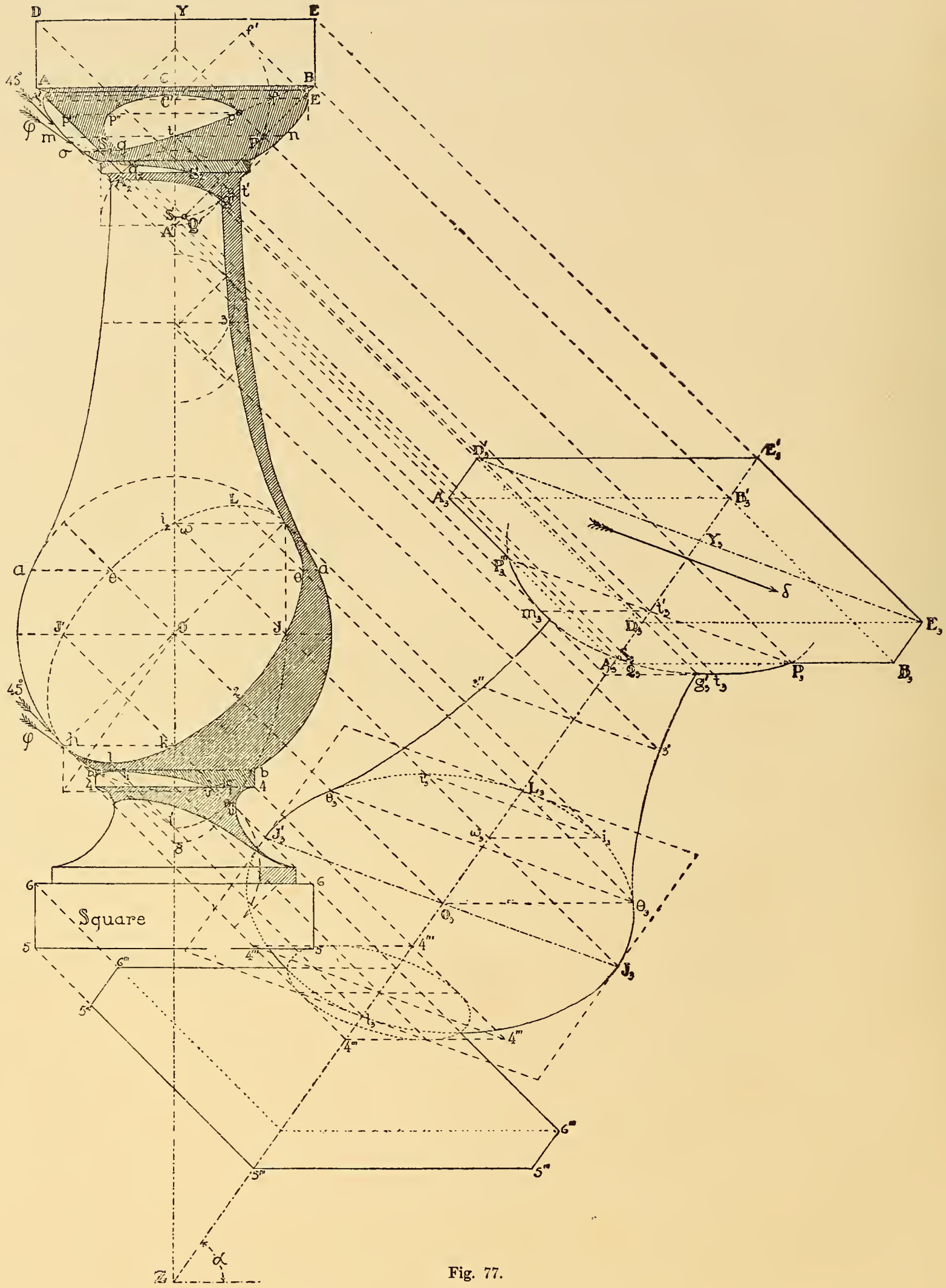

Fig. 77.

cylinder in whose surface the circle of the head of the flute is found, this cylinder will be horizontal and at 45°, or parallel in plan to the horizontal projection of the ray of light, and its shade will pass through the springing points k,k.

(b) In the flute at the right, symmetrical with W′, the analogous cylinder would be perpendicular, in plan, to the ray of light, and its shade will be found by drawing a tangent at the angle φ to the circle which forms the head of the flute, or, what is the same thing, a 45° tangent to the ellipse m, which is the vertical projection of that circle.

(c) In the middle flute the point n is the 45° point of tangency (niche shadow).

(d) The outline of B, the last flute on the right, has the same orientation with reference to the light as that of C on the left; the flutes A and C, owing to their symmetrical position with reference to the light, have the same shadow; hence we may conclude that the theoretical point n′ at the left will be on the same level with n of the middle flute. This gives four points of departure, k,n,m,n′. Join these by a continuous curve; this curve gives, at its intersections with the other ellipses, the other points of departure.

The remainder of the drawing is easily finished.

(D) Notes.—In order to avoid confusion on the drawing we have not shown the shadows cast by the spirals of the volute into the channels which separate the spirals. In order to draw this shadow we shall suppose, in our first approximation, that these channels are all in one plane, which reduces the problem to this: The shadow of a plane figure on a plane parallel with it. The shadow will be exactly the same as the figure itself. Displace the volute at 45° to the right and downward, and produce the shadow. Afterward, by the use of the imagination, for geometry does not serve here, we may increase or decrease the depth of this preliminary shadow according as the real depth of the channel is greater or less than that considered provisionally in the first place.

§ 44. Shadows of a Baluster. (Fig. 77.)

(Problem given at the Ecole des Beaux Arts, March 6th, 1886; to be done *en loge*, on a half double elephant sheet.)

Statement.—The drawing herewith gives, half size, the approximate profile of the baluster. The swell aa,bb is exactly spherical. The remainder of the profile is drawn freehand so as to avoid angles in the line of shade.

It is required: 1st. To determine the shades and shadows on the baluster itself; 2d. To draw the shadows on a roof parallel with the ground line and inclined to the horizontal plane at the angle $\propto$, which shall be an angle between 50° and 60°. The point Z is where the axis of the baluster, if prolonged, would pierce the roof.

Solution of the problem:

I. Shadows on the baluster. (a) Echinus. Find the shade of the echinus as in the case of the torus (§ 18) and of the Doric capital (§ 37). A 45° tangent gives at m the point on the contour. t, on the axis, is in the same horizontal with m. The point f on the equator is found by the two 45° lines C′f′ and Ef′ and the arc of a circle.

(b) Shadow of the abacus on the echinus. Find, 1st, the oval curve of the echinus in fP′g′S′; 2d, in BA′ (at 45°) the auxiliary shadow of the edge of the abacus on the 45° oblique plane. The points P′ and Q′, where these two lines cross, give, by passing back at 45°, the two points of loss p and q.

For the remainder of the shadows of the echinus we may refer to the shadows of the Doric capital (§ 37).

(c) Shade on the swell. This is the shade of a sphere. Find in 1′J the oval curve of the sphere, and in 8v′10 the auxiliary shadow of the base of fillet No. 1.

From v′, which is the intersection of these two lines, by passing back at 45°, derive the point of loss on the shadow of the sphere on fillet No. 1.

(d) Scotia. (See § 40.)

II. Shadow on the roof. (a) The axis. By drawing from Z, foot of the axis on the roof, a line inclined at the angle $\propto$ (the angle of the roof) we have in ZE'_3 the shadow cast by the axis.

(b) The abacus. It will be remembered (see § 11) that such corners of horizontal squares as are located

in the 45° plane of symmetry cast shadows which coincide with the shadow of the axis (as in $E'_3B'_3D_35'''$, etc.); this enables us to find, by lines at 45°, the parallelograms of shadow cast by these squares.

Draw the diagonal $E_3D'_3$ of one of these parallelograms; let δ represent its direction; this gives the shadow of the diagonal of the square; all these diagonals, being in the 45° oblique plane and facing the light, will cast the shadows of their extremities at equal distances on each side of the shadow of the axis; hence, having found such points of shadow as $P_3g'_3t_3\theta_3$, etc., located at one side of the shadow of the axis, we shall be able to locate other corresponding points at equal distances on the other side of the shadow of the axis.

In short, the shadow of the axis is bisectrix of all the parallels of the direction δ.

(c) The echinus. The lowest point S of the separatrix casts shadow in S_3 on the shadow of the axis (tangent to δ). The points of loss p and q cast shadows in P_3 and Q_3 on the shadow of the abacus A_3B_3. The point m on the contour casts shadow in m_3 (tangent at 45°) and t in t_3 (tangent horizontal).

This gives seven points of the shadow of the echinus, and is sufficient for drawing the curve.

By casting from g', the point of loss, to g'_3, we find a point of the shadow of the body of the baluster.

(d) The sphere. Find the shadows cast by the two conjugate diameters JJ' (points on the equator) and lL (highest and lowest points) of the separatrix, and we find in $J_3J'_3$ and l_3L_3 two conjugate diameters of the ellipse of shadow; by the aid of these we draw a circumscribed parallelogram. This is done as follows: 1st. The points l and L, being in the 45° plane of symmetry, cast shadows on the shadow of the axis; 2d, The points J and J', being 45° points on the equator, cast shadow in J_3 and J'_3 on the parallel of direction δ drawn through the shadow of the centre of the sphere O_3.

The remainder of the construction is easily understood.

PROBLEMS FOR SOLUTION.

§ 45. SPHERES—PYRAMIDS.

(Exercises given at the École des Ponts et Chaussées, Preparatory Course.)

It is required to determine the shades and shadows of the geometric figure here shown, the ray of light being the empirical ray at 45°.

I. Sphere intersected by three cylinders. (Fig. 78.)

A sphere has six equal cylindrical tenons at right angles with one another. The tenon which is entirely hidden in the elevation is affixed, by its base, to a vertical plane.

Determine—1st. The shades on the sphere and on the cylinders; 2d. The shadows of the whole body on the vertical plane; 3d. The shadows of the tenons on the sphere and of the sphere on the tenons.

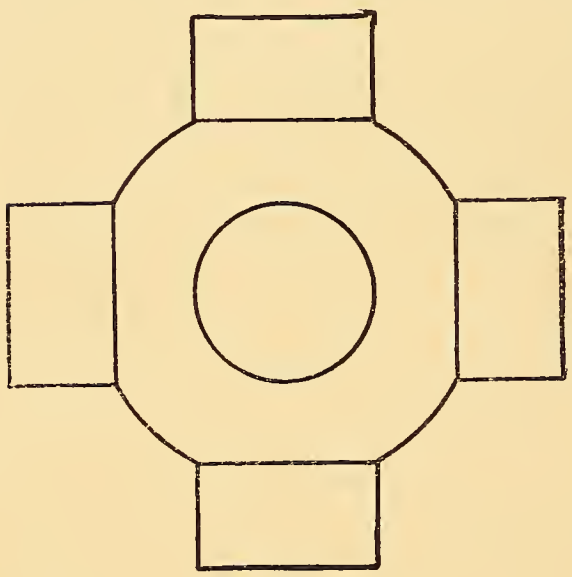

Fig. 78.

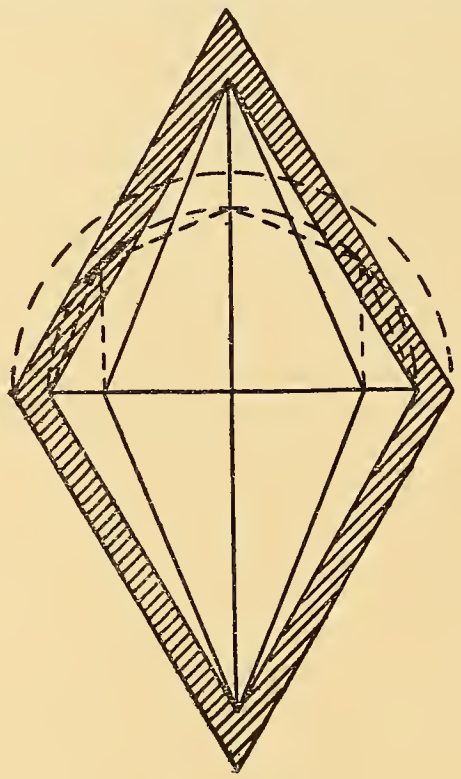

Fig. 79.

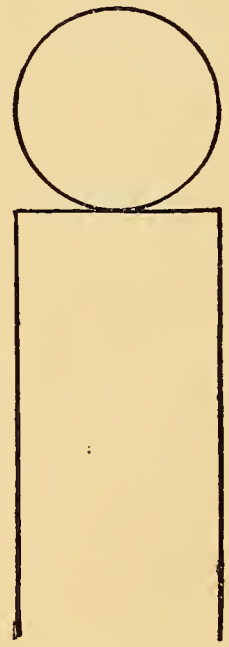

Fig. 80.

II. A double hollow pyramid, in section. (Fig. 79.)

This double pyramid is represented in section. The common base is half a regular octagon, as shown. The figure stands a short distance in front of a vertical plane.

It is required: 1st. To determine the shadow on the vertical plane; 2d. To draw the shadow in the interior of the double pyramid.

III. A sphere and cylinder casting shadow on a roof of 30° pitch and parallel with the ground line. (Fig. 80.)

A cylinder of revolution supports a sphere of the same radius.

(a) Choose a point where the axis pierces the roof and find the curve in which the cylinder intersects the roof.

(b) Find shade on sphere and cylinder.

(c) Cast shadows on roof.

§ 46. A SHORT COLUMN WITH CAP OF BYZANTINE FORM. (Fig. 81.)

(Problem given at the École des Beaux Arts, to be done *en loge;* time for examination, 7 hours; half double elephant sheet.)

Statement.—The drawing herewith gives, at $\frac{1}{3}$ the size that it is to be drawn, an outline of the motive of architecture whose shadows are to be drawn.

The cylindrical shaft carries a portion of a sphere, which is intersected by four half cylinders.

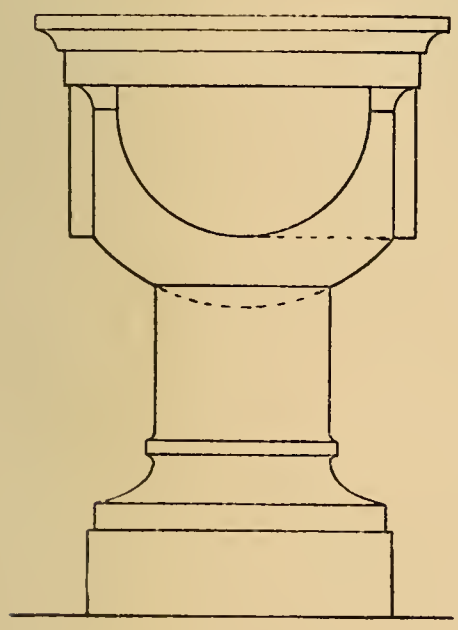

Fig. 81.

A square abacus with a cavetto crowns the whole. In order to avoid any possibility of misunderstanding the form, we have given a perspective detail of one angle.

It is required: 1st. To determine the shades of the solid; 2d. The shadows on a roof parallel with the ground line, inclined at 35° from the horizontal, and placed in such a way that the axis of the motive, prolonged, cuts the roof at a distance below the base equal to $\frac{1}{3}$ of the total height.

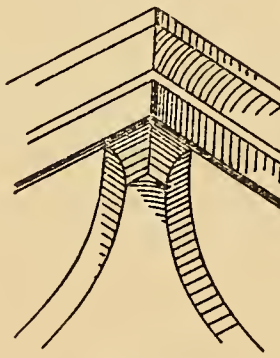

Fig. 81 bis.

§ 47. A SHAFT WITH BASE AND MODILLIONED CAP. (Fig. 82.)

(Problem given at the École des Beaux Arts, to be done *en loge* in 7 hours; half double elephant sheet.)

Statement.—The drawing herewith gives, at half the size to be drawn, a sketch of the motive of architecture of which a drawing is to be made and the shadows cast.

The circular corona T is supported by twelve modillions, whose profile in elevation is given at Q′ and whose breadth in plan is given at Q.

The motive is supposed to project above a roof that is parallel with the ground line and inclined at the angle shown at $\propto$. The lower line of the plinth is a horizontal in the roof.

It is required: 1st. To put in projection the modillions and all the motive; 2d. To find the shadows on the motive as well as on the roof at the back. The ray of light is the empirical ray at 45°.

We shall leave it to the reader to execute this drawing without detailed explanations. It is an application of the shadow of the circular fillet (§ 21 and § 22) combined with the "interruption of shadows." For example:

1st. Find the shadow of the circle CC′ on the face of the modillions, the latter considered as a complete cylinder, without interruptions between the modillions. (Shadow of a circular cap.)

2d. Find the shadow of the same circle CC′ on the shaft.

3d. Determine the shadow of the circle D (uninterrupted), first, on the shaft, and, second, on the cavetto, whose profile is shown at DE; also supposing this cavetto uninterrupted between the modillions. (Shadow of a circular fillet with conjć, § 22.)

4th. Find the shade (theoretical) of this same cavetto, and the points, such as m, where it cuts the left

side of the modillions, are the points of origin of such shadows as mn′; these are entirely analogous to the bridge shadow (§ 26).

All these general shadows once determined, the interruptions, such as 2, 3, 4, etc., of these, as well as their offsets and points of change, are easily derived.

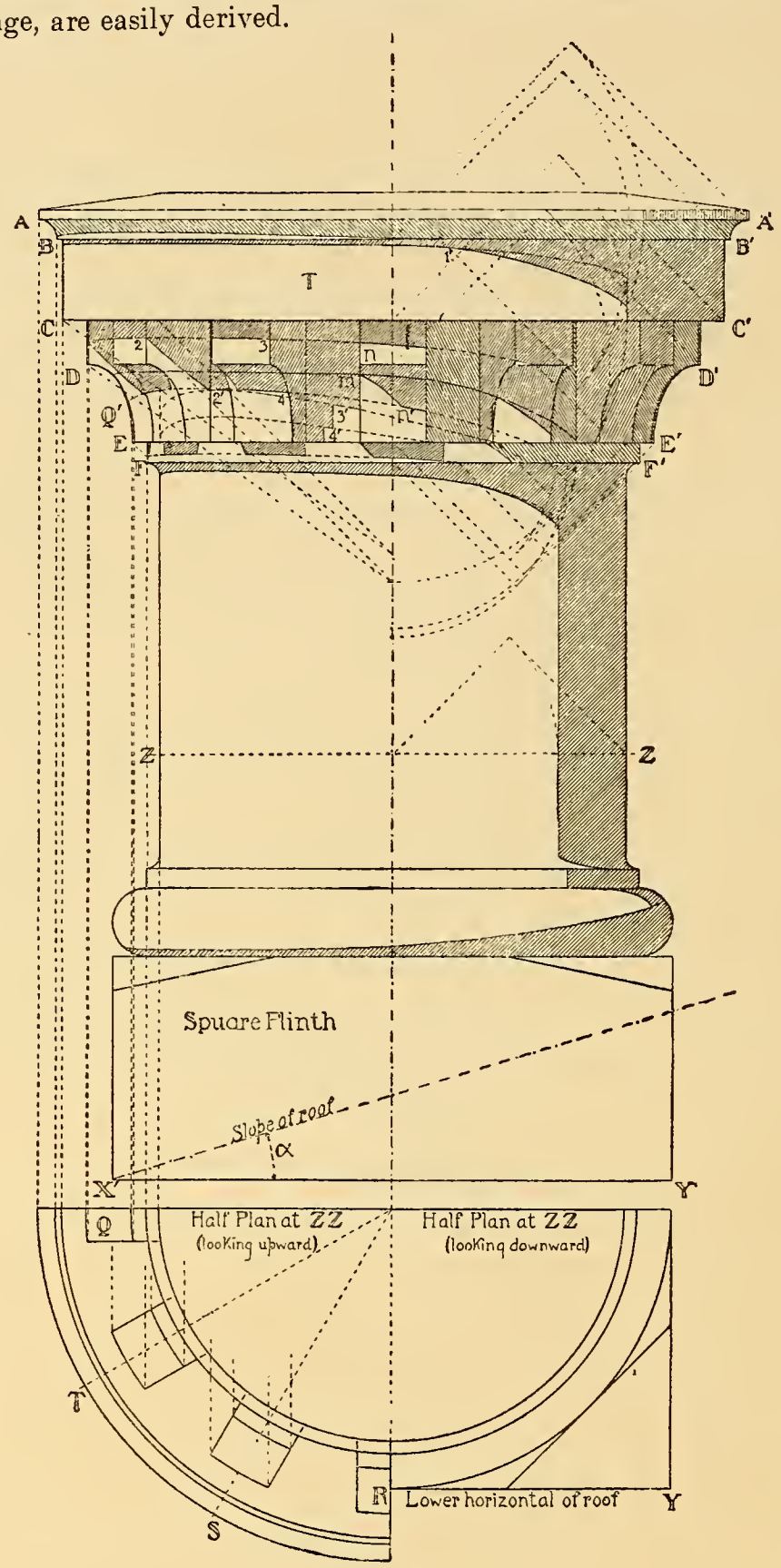

Fig. 82.

The drawing herewith gives a fairly correct idea of the shadow; but the scale is too small to permit of great accuracy.

§ 48. Figures 83, 84, 85 and 86 are familiar problems in Shades and Shadows, and have been used for class exercises and examinations in the Department of Architecture at the University of Pennsylvania.

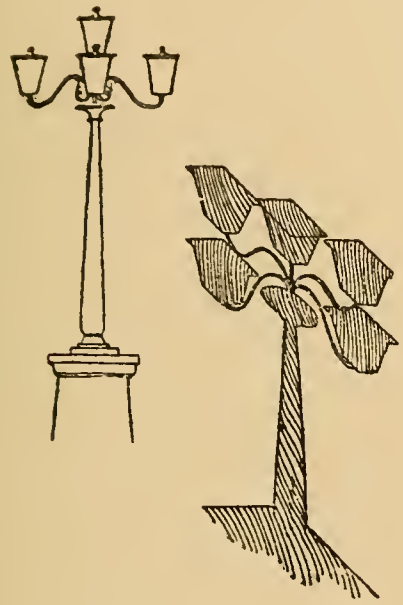

Fig. 83.

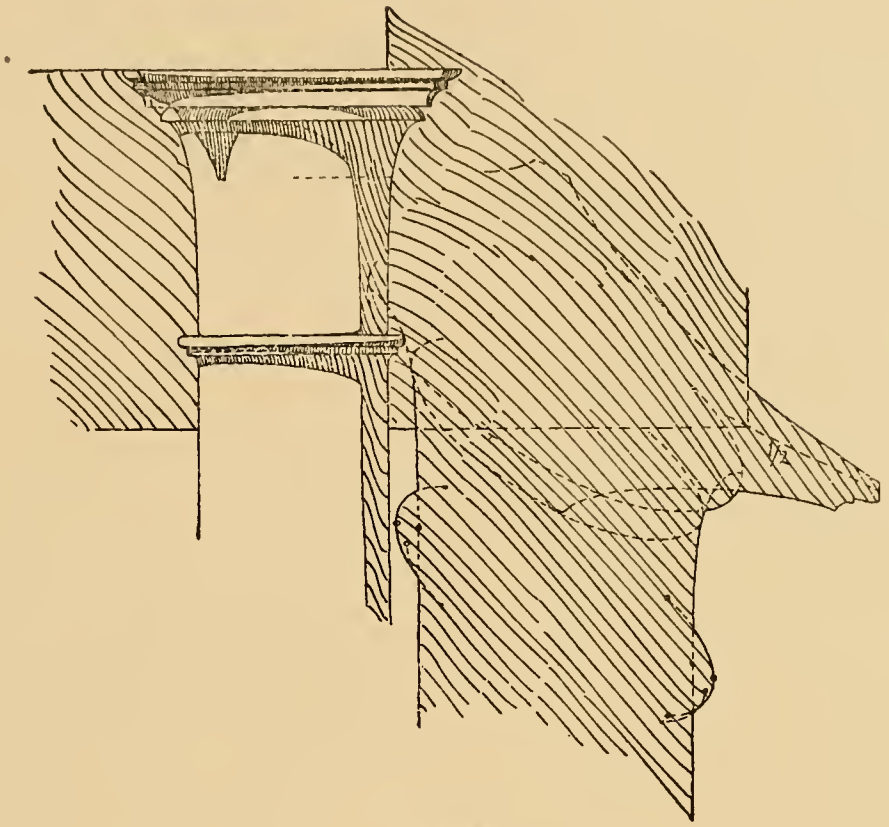

Fig. 85.

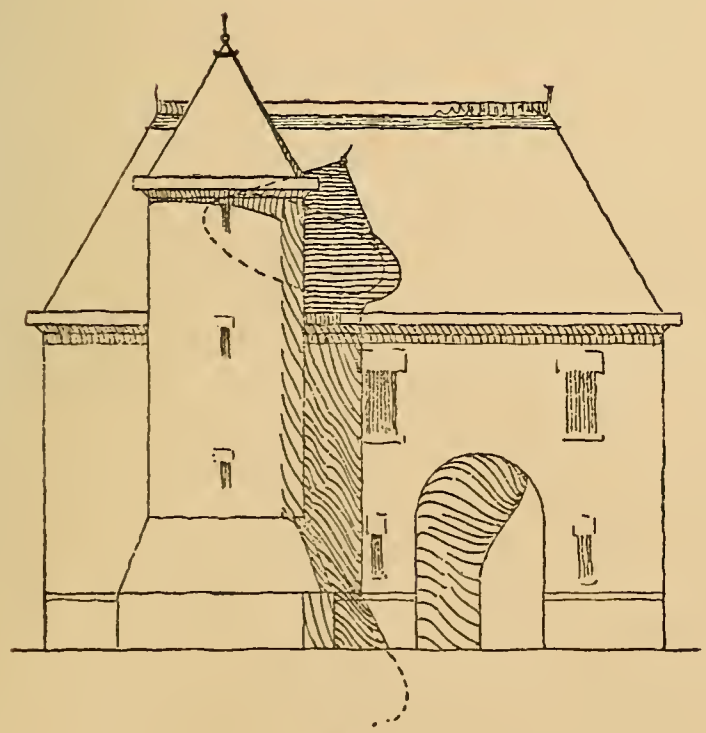

Fig. 86.

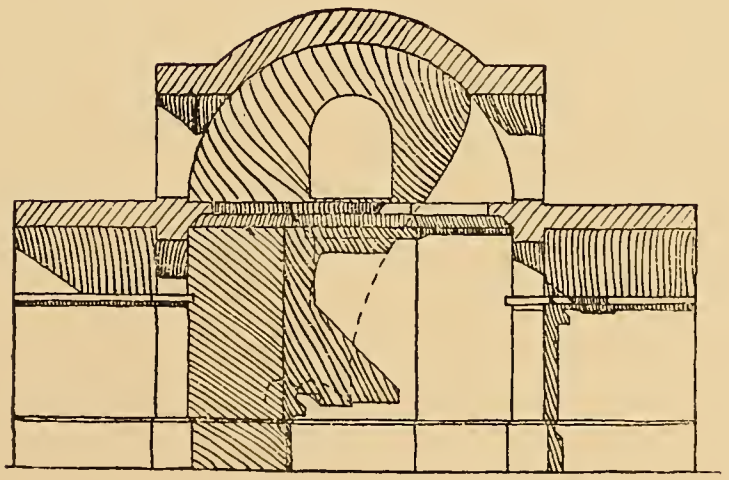

Fig. 84.

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